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TESIS DOCTORAL

ESSAYS ON MICROECONOMICS

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DEPARTAMENTO DE ECONOMÍA

Getafe, diciembre de 2013

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Contents

Abstract

This thesis discusses three cases related to microeconomics in dynamic frameworks. It consists on the following chapters:

First chapter studies the optimal selling mechanism for a seller who puts up for sale one individual unit per period to a single buyer in a two-period game. The buyer's willingness to pay remains constant over time and is his private information. In the first period, the seller can commit to a mechanism for the current period but not for the second one. The main result is that the seller cannot achieve greater payoffs than those obtained by posting a price in each period. However, price posting is not optimal if the buyer is sufficiently impatient relative to the seller. Finally, it is shown that a mechanism à la Goethe (see Moldovanu and Tieztel 1998) is almost optimal.

Second chapter studies the previous model in a multi-period setting when seller and buyer are equally patient. In the two-period model, the degree of that patience did not affect the set of feasible mechanisms. However, with more than two periods, a larger patience of players do affects this set. In particular, some price posting mechanisms that were optimal when players were impatient are not longer feasible when they are sufficiently patient. Additionally, with more than two periods, the seller could engage in gradual learning. The main result is that a seller cannot do better than posting a price in every period. There is also a complete characterization of the optimal mechanism and equilibrium payoffs for every prior. Finally, it shows that when seller and buyer are arbitrarily patient, the seller does not learn about buyer's type except in extreme cases, posting a price equal to the minimum buyer's willingness to pay in every period. This result is a reminiscence of the Coase's conjecture, where a monopolist cannot exert her monopoly power due to the lack of long-term commitment.

Third and last chapter proposes a model where firms, which compete for high-skill workers, can distort their production in order to conceal information to the market about the skill of their workers. This occur in a framework where firms differ in their marginal labor productivity and workers in their skill. The main result is that firms actually distort their production and that these distortions are not monotonic in the marginal labor productivity.

Resumen

Esta tesis discute tres casos relacionados a la microeconomía en entornos dinámicos estructurada de la siguiente manera:

El primer capítulo estudia el mecanismo de venta óptimo de un vendedor que vende una sola unidad por período a un único comprador en un juego de dos períodos. La disponibilidad a pagar de este comprador se mantiene constante en el tiempo y es su información privada. En el primer período, el vendedor se puede comprometer a un determinado mecanismo para dicho período pero no para el siguiente. El resultado principal es que el vendedor no logra obtener mejor rentabilidad que la obtenida mediante el uso de un precio en cada período. Sin embargo, usar un precio no es óptimo si el comprador es lo suficientemente impaciente en comparación al vendedor. Finalmente, se muestra que un mecanismo 'à la Goethe' (ver Moldovanu y Tieztel 1998) es cuasi-óptimo.

El segundo capítulo estudia el modelo anterior en un entorno con mucho períodos, siendo el vendedor y el comprador igualmente pacientes. En el modelo de dos períodos, el grado de paciencia no afectaba el conjunto de mecanismos factibles. Sin embargo, con más de dos períodos, una mayor paciencia de los jugadores sí afecta dicho conjunto. En particular, algunos mecanismos de precios que son óptimos cuando los jugadores son impacientes ya no son viables cuando son los suficientemente pacientes. Adicionalmente, con más de dos periodos, el vendedor podría emplear un proceso de aprendizaje gradual. El resultado principal es que el vendedor no puede hacerlo mejor que cuando usa un precio en cada período. Se incluye una caracterización completa del mecanismo óptimo y de los pagos de equilibrio para cada creencia a priori. Finalmente se muestra que, cuando el vendedor y el comprador son arbitrariamente pacientes, el vendedor no logra aprender sobre la disponibilidad a pagar del comprador excepto en casos extremos, usando en cada período un precio igual a su mínima disponibilidad de pago. Este resultado es reminiscente de la conjetura de Coase, en el cual un monopolista no puede ejercer su poder de monopolio debido a la falta de compromiso a largo plazo.

El tercer y último capítulo propone un modelo en el cual las firmas, que compiten por trabajadores altamente cualificados, pueden distorsionar su producción con el objetivo de ocultar información al mercado sobre las cualificaciones de sus trabajadores. Esto ocurre en un entorno en el cual las firmas difieren en su productividad marginal del trabajo y los trabajadores en sus cualificaciones. Se muestra que dichas distorsiones ocurren y que no son monótonas en la productividad marginal del trabajo.

Agradecimientos

Esta tesis ha sido posible gracias a la colaboración invaluable y desinteresada de muchas personas, entre los que se encuentran profesionales del área académica y amigos. Por ello me gustaría desde aquí, ofrecerles un sincero agradecimiento a todos ellos:

Agredezco profundamente a mi supervisor Ángel Hernando-Veciana por todo su apoyo, consejos y enseñazas. Por su rigurosidad y por sacar siempre lo mejor de mi. Pero por sobre todo, por su paciencia y dedicación. Han sido años durísimos pero han valido la pena.

A Matilde Machado, por estar siempre. Por sus continuas palabras de ánimo. Por todos sus consejos. Por su optimismo y por creer siempre en mi.

No sé si esta tesis hubiese sido posible sin Ángel y Matilde. No tengo palabras suficientes para ambos. Gracias!

A Marco Celentani, Mikhail Drugov y Antoine Loeper quienes, con todos sus comentarios y sugerencias sobre mi investigación, me han sido de gran ayuda para terminar esta tesis. Gracias tambien por sus comentarios a Antonio Cabrales, a Diego Moreno y en especial, junto con un gran recuerdo, a María Ángeles de Frutos. Un ejemplo a seguir.

A Eugenio Giolito y Carlos Ponce, también por sus consejos y por preocuparse tanto. Por las buenas charlas que teníamos, que se extrañan.

A mis amigos y compañeros de ruta que han sido la mejor parte de todos estos años: Daniel "Dani" Garcia, Luis "el Percha" Aguiar, Luis "Luifa" Franjo, Lian "Lían" Allub, Carlos "PBT" Perez, Joaquin "Jota" Coleff, Dolo de la Mata, Rama de Elejalde, Marianito y Fer Scapin, Juan y Diego, Zoe y Gregorio, Lucas Cambiano, Agustín "el Pulpo" Riveiro, y el quinteto mayor (Kili-Ruso-Cesar-Bbto-Cuadra'o). A todos ellos, gracias por los buenos momentos (y también por los malos, por qué no) y por el aguante.

A mi familia, que a pesar de la distancia, siempre han estado conmigo, con todo su apoyo y comprensión. "Yo he viajado hasta acá para decir que la familia es importante y siempre está", dijo Camila el 29 de Octubre del 2013.

A Gini y Gonza por ser mi familia adoptiva madrileña.

Y en especial, a Tati, que me ha sufrido (sin chistar) todo este tiempo. Gracias Tatin, por estar.

Como no podía ser de otra forma, porque sino no serían mios, estos agradecimientos han sido cortos y directos. Aunque seguramente (lo lamento, no me sale de otra forma) muy parcos. Para ver ejemplos de agradecimientos bien escritos, divertidos y a su vez emotivos, ver las tesis de Dani Garcia (2012) y Luifa Franjo (2013) (no están incluídas en la bibliografía pero se pueden encontrar en el siguiente enlace: http://portal.uc3m.es/portal/page/portal/postgrado_mast_doct/tesis_doctoral).

Para Jorge.

Chapter 1

Optimal Selling Mechanism in a Repeated Game under Imperfect Commitment: The Two-Period Case

1.1 Introduction

In 1797, Goethe was in the process of trying to sell his most recent work, the epic poem Hermann and Dorothea. However, he was concerned about the information asymmetry between him and the publisher with respect to the publisher's valuation of his work.[1] Goethe decided to propose the following selling mechanism: each one (Goethe and the publisher) would send a sealed note with their demanded price to a lawyer; the sale would take place at Goethe's price only in the case that the publisher's demanded price was higher than or equal to Goethe's demanded price.[2] With this mechanism, Goethe wanted to learn something about the publisher's valuation and obtain some advantage in future transactions.

Goethe's story illustrates a common situation in the market place. A seller wants to sell something to a buyer whose willingness to pay is private. However, the seller may use information from past sales to the same buyer to infer his willingness to pay. This is reminiscent of the problem that firms currently face when trying to record information about individual consumer behavior. Due to new technologies such as online purchases or fidelity programs, it seems a pervasive problem. With these technologies sellers want to learn consumer preferences. Tesco, the largest retailer in Britain demonstrates a good example of these learning attempts through a fidelity program.[3] This is currently a practice that is becoming the norm in retail and is not restricted to large chains. Zen Nippon Shokuhin, a small grocery club in Japan, follows Tesco's example: it collects and analyzes data from its customers to learn their preferences.[4]

[1] See Moldovanu, B. and Tieztel, M. (1998) for the complete story.

[2] As Moldovanu and Tietzel (1998) pointed this is a second-price auction in which the sealed reserve price of the seller has the effect of a second bidder.

[3] See The Economist, (2005).

[4] See The Economist, (2011).

On principle, knowledge of consumer preferences can be used by firms to implement pricing schemes that better discriminate among consumers. However, from a theoretical point of view, this is not obvious because consumers (and the publisher in Goethe's case) may have incentives to act strategically to mislead the learning process of the seller.

For instance, Hart and Tirole (1988) argue that a monopolist that sells a perishable unit in each period and has full commitment power at the beginning of the game finds it optimal to commit to ignoring all the information that she learns along the equilibrium path about the type of the buyer. Then, the buyer has no incentives to lie during one period to manipulate the seller's belief. However, full commitment is a very extreme assumption. In a long-term relationship the seller has to specify in the contract all potential contigencies during the complete timespan. However, these contingencies (such as new technologies) could be difficult to foresee. Additionally, the long-term contract must resist all possible renegotiations due to ex-post inefficiencies that usually arise in asymmetric information frameworks. For this reason, long-term contracts are very difficult to write in the real world. In fact, it is possible to find many situations in which a short-term commitment relationships fits better.[5]

There is extensive literature related to bargaining under conditions of asymmetric information where only one party has the right to make offers. Most of them study the case of durable goods, in which the game finishes when a buyer accepts an offer.[6] On the other hand, Hart and Tirole (1988) and Schmidt (1992) study the case of repeated bargaining where a player with bargaining power trades a service or perishable good in every period with non-anonymous and sufficiently patient agents.[7] All these previous articles restrict the monopolist's strategy to a sequence of posted prices. Nothing guarantees that this mechanism is the optimal one. It is natural to ask if the monopolist has a better selling mechanism to maximize her benefits.

The purpose of this paper is to study the conditions under which price posting may be an optimal selling mechanism. We also show that it is possible to rationalize the mechanism used by Goethe. Although Moldovanu and Tieztel (1998) shows that Goethe's mechanism is optimal in an static framework, the description of the story fits better with our dynamic framework. In our framework, Goethe's mechanism is almost optimal if one assumes that the publisher is relatively impatient with respect to Goethe. This seems a reasonable assumption since publishers could not count on dealing with Goethe in the future with any type of certainty.

We consider the case in which a seller commits to use a selling mechanism for the current period, but not for future ones. We study this problem in a two-period model with one seller and one buyer. The buyer has two possible valuations for the good which are his private information. In every period the seller has one perishable good to sell, which is produced at zero cost. The seller can propose a different selling mechanism in every period.

Skreta (2006) has shown that posting a price is the optimal selling mechanism when a monopolist with a short-term commitment has a durable good to sell to a single buyer that the monopolist addresses repeatedly. In this paper, we look for the optimal selling mechanism when such a monopolist sells instead a perishable good or a service as studied by Hart and Tirole (1988) (in their renting framework) and by Schmidt (1992). Technically, there is a crucial difference. In Skreta (2006), the

[5] For some real world examples about the inability of the principal to commit see Laffont and Tirole (1988), Laffont and Tirole (1993) or McAfee and Vincent (1997).

[6] See for example Fudenberg, Levine and Tirole (1985), or Sobel and Takahashi (1983).

[7] Hart and Tirole study both cases: the durable good case and the case in which the monopolist decides to rent it.

game finishes when the buyer buys the good, but this is not the case in our model. Her procedure is not directly applicable to our framework because she sustains her analysis on the fact that the only non-trivial continuation value arises when the good is not sold. In our model, the buyer has to take into account how his future surplus is going to be affected in case of buying and in case of rejecting the good, i.e. there are two continuation values.

To resolve the model, we use a dynamic mechanism design approach following the procedure proposed in Bester and Strausz (2001). In that article, they provide a modified version of the revelation principle where the seller has imperfect commitment.

The rest of the paper is organized as follows. Section 2 provides a general setup of the problem and review the Bester and Strausz (2001) revelation principle for this type of environment. Section 3 analyzes the problem and gives a characterization of the optimal selling mechanism. Section 4 illustrates, by example, that this result does not hold when players have different discount factors, and shows that the mechanism proposed by Goethe can be interpreted in this direction. Finally, Section 5 concludes. Those proofs considered relevant for the general understanding of the model are included in the main text while the rest can be found in the Appendix.

1.2 General Setup

Next, we propose a dynamic problem that follows the framework proposed by Bester and Strausz (2001). The problem is solved by recursive methods as they suggest. Therefore, in this section we directly propose a dynamic problem as a sequence of static problems. We show in the Appendix how our recursive formulation corresponds to the sequential two-period problem.

We consider a two-period game with $r = \{1, 2\}$, where r is the number of periods remaining at the beginning of the current period. There is one risk-neutral seller (the principal) and one risk-neutral buyer (the agent) facing each other repeatedly. Both players discount the future at the same rate $\delta \in (0, 1]$. At every period, the seller can produce at zero cost a non-storable object that is put up for sale to the buyer.[8] This buyer has valuation θ_i for the good, where $\theta_i \in \Theta = \{\theta_L, \theta_H\}$. We call θ_L (θ_H) the low-type buyer (high-type buyer) and sometimes we denote it by the subscript L (H). This valuation remains constant over time and is the buyer's private information. The probability of a high-type buyer is denoted by $p_{H,3}$, and for a low-type buyer by $p_{L,3} = 1 - p_{H,3}$. We refer to this as the prior of the seller.

A mechanism Γ_r in period r specifies a message set M_r and a decision function $y_r = (x_r, w_r)$, where $x_r : M_r \to [0, 1]$ is the allocation rule and $w_r : M_r \to \mathbb{R}$ is the payment rule. Then, each element $m_r \in M_r$ commits the seller to implement the allocation rule $x_r(m_r)$ and requires for the buyer the payment $w_r(m_r)$.

The seller has imperfect commitment. This is, during the first period the seller can commit herself to a mechanism for the current period but not to a mechanism for the next period. So, at the beginning of period $r = 2$ the seller chooses a mechanism $\Gamma_2 \in \Upsilon$ given her prior $p_{H,3}$ about facing a high-type, where Υ is the space of mechanisms. Next, the buyer observes Γ_2. His strategy specifies the probability $q_i(m_2)$ with which the buyer sends each message m_2, where $q_i : M_2 \to [0, 1]$, for $i \in \{L, H\}$ and that verifies $\sum_{m_2 \in M_2} q_i(m_2) = 1$. The buyer can always choose not to participate

[8] All our results hold for any constant production cost strictly less than the minimum possible value that the buyer is willing to pay.

in the mechanism Γ_2.[9] In this case, he gets zero instant payoffs but he can choose to participate in the second period. Next, the seller observes m_2, implements the mechanism and updates her beliefs about facing a high-type buyer. We denote it by $p_{H,2}(m_2)$ and it is updated following a mapping $p_{H,2} : M_2 \to [0,1]$. In the following, we use $p_{L,2}(m_2)$ to indicate $1 - p_{H,2}(m_2)$ and $p_2(m_2)$ to indicate the vector of posteriors $(p_{L,2}(m_2), p_{H,2}(m_2))$ when a message m_2 is sent. Updated beliefs constitute the state variable for the next period. Then, at the beginning of period $r = 1$ the seller chooses a new mechanism $\Gamma_1 \in \Upsilon$ given her updated beliefs, and the buyer observes Γ_1 and chooses his strategy in response. The seller observes m_1, implements the mechanism Γ_1 and the game finishes.

We denote by $v_r(m_r)$ and $u_{i,r}(m_r)$ to the seller's and buyer's *instant* payoff, respectively, when the buyer with valuation θ_i sends the message m_r, i.e.

$$v_r(m_r) = w_r(m_r),$$
$$u_{i,r}(m_r) = x_r(m_r)\theta_i - w_r(m_r).$$

Let $V_1 : [0,1]^2 \to \mathbb{R}$ and $U_{i,1} : [0,1]^2 \to \mathbb{R}$ represent the continuation values for each player when $r = 2$.[10]

Consequently, given a prior $p_3 \equiv (p_{L,3}, p_{H,3})$, the seller's problem at period $r = 2$ is to choose (q_2, p_2, Γ_2) that maximize:[11]

$$\sum_{i\in\Theta} \sum_{m_2\in M_2} p_{i,3} q_i \left(m_2\right) \left(v_2(m_2) + \delta V_1 \left(p_2(m_2)\right)\right), \tag{1.1}$$

where $q_2 \equiv \left(q_2\left(m_2\right)\right)_{m_2\in M_2}$, $q_2\left(m_2\right)$ indicates the vector $(q_L(m_2), q_H(m_2))$, and $p_2 \equiv \left(p_2\left(m_2\right)\right)_{m_2\in M_2}$, is subject to the following constraints:

- The buyer's Incentive Compatibility $(IC_{i,2})$: the buyer chooses his optimal reporting strategy, i.e.,

$$\sum_{m_2\in M_2} q_i\left(m_2\right)\left(u_{i,2}(m_2) + \delta U_{i,1}\left(p_2\left(m_2\right)\right)\right) \geq \tag{1.2}$$
$$\sum_{m_2\in M_2} q_i'\left(m_2\right)\left(u_{i,2}(m_2) + \delta U_{i,1}\left(p_2\left(m_2\right)\right)\right)$$

  for $i \in \{L, H\}$, and for all $q_i'\left(m_2\right)$.

- The buyer's Individual Rationality $(IR_{i,2})$: The buyer's individual rationality constraint has

[9] Note that our definition of the mechanism requires participation. We take the usual convention that the buyer can decide whether to participate or not, getting zero payoffs in the last case. This convention is discussed later, when we talk about the individual rationality constraint (IR). Alternatively, it is possible to include a message in M_2 that represents no participation.

[10] Continuation values depends on the vector of priors at the beginning of the period. Since there are two types, the vector of priors is completely determined by the prior about facing a high type, i.e. $p_{H,r+1}$. Then, later in the paper, and with some abuse of notation, continuation values will be represented as depending only in that prior.

[11] In Bester and Strausz speficifaction, it is allowed $v_{i,2}(m_2) \neq v_{j,2}(m_2)$ when $i \neq j$ giving $\sum_{i\in\Theta}\sum_{m_2\in M_2} p_{i,3}q_{i,2}\left(m_2\right)\left(v_{i,2}(m_2) + \delta V_{i,1}(p_2)\right)$.

to be satisfied for all types to which the seller assigns positive probability:

$$p_{i,3}\left[\sum_{m_2\in M_2} q_i\left(m_2\right)\left(u_{i,2}(m_2)+\delta U_{i,1}\left(p_2\left(m_2\right)\right)\right)-\delta\bar{U}_{i,1}\right]\geq 0 \tag{1.3}$$

for $i\in\{L,H\}$, where $\bar{U}_{i,1}$ is the continuation value when the buyer choose not to participate in the mechanism Γ_2. Although there is no loss of generality in assuming that the buyer participates with probability one, we have to warranty that he does not do better staying out. This implies $\bar{U}_{i,1}\geq 0$. We assume $\bar{U}_{i,1}=0$ since it is the less restrictive in (1.3) and, as we will show later, this is the case at the optimal contract (given we can assume any belief for the out-of-equilibirum message).

- And finally, for each message, the seller's updated belief $p_{i,2}\left(m_2\right)$ has to be consistent with Bayes' rule (BR_2) whenever possible:

$$p_{i,2}\left(m_2\right)\sum_{j\in\Theta} p_{j,3}q_j\left(m_2\right)=p_{i,3}q_i\left(m_2\right). \tag{1.4}$$

It follows that the seller's problem with imperfect commitment at $r=2$ is given by:

$$V_2(p_3)=\underset{\{q_2,p_2,\Gamma_2\}}{Max}\sum_{i\in\Theta}\sum_{m_2\in M_2} p_{i,3}q_i\left(m_2\right)\left(v_2(m_2)+\delta V_1\left(p_2\left(m_2\right)\right)\right), \tag{1.5}$$

subject to $(1.2)-(1.4)$.

We say that the outcome (q_2,p_2,Γ_2) is *incentive feasible* if it satisfies (1.2)-(1.4) for all $\theta_i\in\Theta$. Additionally, it is *incentive efficient* if it satisfies (1.5), the seller chooses the best outcome among all of the incentive feasible ones. An *optimal mechanism* is a mechanism Γ_2 that belongs to an incentive efficient outcome (q_2,p_2,Γ_2). Finally, (q_2,p_2,Γ_2) and (q_2',p_2',Γ_2') are *payoffs equivalent* if they leave the seller and the buyer (of every possible type) with the same payoffs, i.e.

$$\sum_{i\in\Theta}\sum_{m_2\in M_2} p_{i,3}q_i\left(m_2\right)\left(v_2(m_2)+\delta V_1\left(p_2\left(m_2\right)\right)\right)=$$
$$\sum_{i\in\Theta}\sum_{m_2'\in M_2'} p_{i,3}q_i\left(m_2'\right)\left(v_2(m_2')+\delta V_1\left(p_2'\left(m_2\right)\right)\right),$$
$$\sum_{m_2\in M_2} q_i\left(m_2\right)\left(u_{i,2}(m_2)+\delta U_{i,1}\left(p_2\left(m_2\right)\right)\right)=$$
$$\sum_{m_2'\in M_2'} q_i'\left(m_2'\right)\left(u_{i,2}(m_2')+\delta U_{i,1}\left(p_2'\left(m_2\right)\right)\right),\quad i\in\{L,H\}.$$

1.2.1 Revelation Principle

In this subsection we show that, i) we can restrict to direct mechanisms, ii) that p_2 is always determined by Bayes' rule (consequently there are not out-of-equilibrium beliefs) and, iii) that it is enough to consider a subset of all possible q_2.

A direct mechanism is a mechanism in which the message set is the buyer's type set, i.e., $M_r = \Theta$. In this case, the buyer's strategy is to report each type with some probability, i.e., $q_i : \Theta \to [0, 1]$, with $\sum_{m_r \in \Theta} q_i(m_r) = 1$. Bester and Strausz (2001) provides a revelation principle for environments with imperfect commitment, including the multistage contracting case as the problem proposed in this work. Based on this revelation principle we can seek for a solution of (1.5) using direct mechanisms, i.e.:

Lemma 1 *Assume a state p_{r+1}. Any solution (q_r, p_r, Γ_r) for (1.5) is payoffs equivalent to a solution $\left(\hat{q}_r, \hat{p}_r, \hat{\Gamma}_r\right)$ where $\hat{\Gamma}_r$ is a direct mechanism and where the buyer reports his type with positive probability, i.e., $\hat{q}_i(i) > 0 \ \forall \ i \in \{L, H\}$.*

Proof. This lemma is a direct application of Proposition 2 and its corollary at Bester and Strausz (2001). ■

Bester and Strausz (2001) shows that it is sufficient for the mechanism designer to consider mechanisms in which the set of messages has equal cardinality to the type space. Moreover, they show that we can associate each message with a type that plays the message with positive probability.

A consequence is that the mechanism designer can be restricted to outcomes (q_r, p_r, Γ_r) where the mechanism has Θ as the message set. Then, as every message that belongs to Θ is reported with positive probability, she can always associate a message with the corresponding type. That is, she asks every type to report the truth with positive probability, i.e., $q_H > 0$ and $1 - q_L > 0$ where we denote by q_H (q_L) the probability that a high-type buyer (low-type buyer) sends a high-type message.

Notice that this revelation principle differs from the standard one (see Myerson 1981) in that there is no guarantee that the buyer reports his true type with certainty. Even so, truthful reporting is always an optimal strategy for the buyer and he still plays it with positive probability.

Given some mechanism Γ_r, (1.2) requires that any message which is played with positive probability must be optimal for the buyer. From the revelation principle either $q_H = 1$ $(q_L = 0)$, in which case (1.2) requires that the high-type (low-type) prefers to report the truth, or $q_H < 1$ $(q_L > 0)$ in which case (1.2) requires indifference between both messages. Hence, in our two period setting, (1.2) can be simplified to:

$$
\begin{aligned}
IC_{H,2}: &\quad u_{H,2}(h) + \delta U_{H,1}(p_2(h)) \geq u_{H,2}(l) + \delta U_{H,1}(p_2(l)) \ \text{ with equality if } 1 - q_H > 0, \\
IC_{L,2}: &\quad u_{L,2}(l) + \delta U_{L,1}(p_2(l)) \geq u_{L,2}(h) + \delta U_{L,1}(p_r(h)) \ \text{ with equality if } q_L > 0,
\end{aligned}
$$

where, from now on, we use h and l to indicate high-type and low-type messages, respectively.[12]

As every message is sent with positive probability by at least one type, (1.4) is always satisfied. As a consequence, the posteriors are completely determined by Bayes' rule and p_2 is a redundant variable of optimization.

[12]Notice that when the buyer is indifferent between both messages, he randomizes between them. The seller knows this but she does not observe which probability the buyer chooses for each message. We assume that she can always select the best equilibrium between all the possible ones as is usual in mechanism design.

Additionally, we can concentrate our analysis on those incentive feasible outcomes such that $q_H \geq q_L$. For those incentive feasible outcomes such that $q_L > q_H$, we can simply define a new mechanism in which the role of each message is interchanged. We prove this in the following lemma.

Lemma 2 *All incentive feasible outcomes (q_r, p_r, Γ_r), where Γ_r is a direct mechanism, such that $q_L > q_H$ is payoff equivalent to an incentive feasible outcome $(\hat{q}_r, \hat{p}_r, \Gamma_r)$, with the same direct mechanism Γ_r, such that $\hat{q}_H > \hat{q}_L$.*

Proof. See the Appendix. ■

Notice that $q_H \geq q_L$ implies $p_{H,2}(h) \geq p_{H,3} \geq p_{H,2}(l)$ by Bayes' rule.

1.3 Optimal Selling Mechanism

In this section we solve the two-period case of the previous problem using backward induction. We prove that the optimal selling mechanism in both periods can be implemented by price posting, i.e., a take-it-or-leave-it offer.

Definition 3 *A Price Posting Mechanisms in period r -in what follows price posting- is an indirect mechanism with two possible messages "$take-it$" or "$leave-it$", with allocation and payment rules according to:*

$$x_r(m_r) = \begin{cases} 1 & if \quad m_r = take-it, \\ 0 & if \quad m_r = leave-it. \end{cases} \quad ; \quad w_r(m_r) = \begin{cases} z_r & if \quad m_r = take-it, \\ 0 & if \quad m_r = leave-it. \end{cases}$$

where $z_r \in \Re$ is the price asked by the seller.

1.3.1 Period r=1

In the last period, instead of using Bester and Strausz, we can use standard mechanism design (see for example Bolton and Dewatripont, 2005) without loss of generality. The seller's problem at $r = 1$ is:

$$\begin{aligned}
&\underset{\{x_1,w_1\}}{Max} \; p_{H,2}v_1(h) + p_{L,2}v_1(l) \quad \text{subject to,} \\
&IC_{H,1}: \;\; u_{H,1}(h) \geq u_{H,1}(l), \\
&IC_{L,1}: \;\; u_{L,1}(l) \geq u_{L,1}(h), \\
&IR_{H,1}: \;\; u_{H,1}(h) \geq 0, \\
&IR_{L,1}: \;\; u_{L,1}(l) \geq 0.
\end{aligned}$$

The set of optimal allocations are,

$$x_1(h) = 1 \;\; \forall \;\; p_{H,2}$$

$$x_1(l) = \begin{cases} 0 & if \quad p_{H,2} > \frac{\theta_L}{\theta_H}, \\ \alpha_1 & if \quad p_{H,2} = \frac{\theta_L}{\theta_H}, \\ 1 & if \quad p_{H,2} < \frac{\theta_L}{\theta_H}, \end{cases}$$

where $\alpha_1 \in [0,1]$. Hence, the set of optimal payments are $w_1(h) = \theta_H - x_1(l)\Delta\theta$ and $w_1(l) = x_1(l)\theta_L$, with $\Delta\theta = (\theta_H - \theta_L)$.

When the seller is optimistic enough in facing a high-type buyer, she proposes a separation mechanism: high-type buyer receives the good with certainty and pays his valuation, while a low-type buyer does not receive the good and pays zero. On the other hand, when the seller is pessimistic she proposes pooling: every buyer gets the good and payment is equal to the low-type valuation. The prior equal to $\frac{\theta_L}{\theta_H}$ is the limit between both mechanisms. We denote it with τ_1^*. At this prior, the seller is indifferent between both mechanisms. She can even propose any mechanism with an allocation for the message l between 0 and 1. However, the seller cannot do better than in the pooling or separation cases. From now on, and to simplify the notation, we are going to consider $\alpha_1 = 0$.[13]

Remark 4 *The optimal mechanism at $r = 1$ can be implemented by a price posting.*

Proof. The proof consists in showing that there is an indirect mechanism with the properties of a take-it-or-leave-it offer that is payoff equivalent to our optimal direct mechanism. Because this is a one period game, it is straightforward. See the Appendix for the details. ■

1.3.2 Period r=2

From previous results we can deduce the continuation values in period $r = 2$ for the high-type buyer, the low-type buyer and the seller, respectively:

$$\begin{aligned} U_{H,1}(p_{H,2}) &= \Delta\theta \mathbf{I}_{\left[0,\frac{\theta_L}{\theta_H}\right)}(p_{H,2}), \\ U_{L,1}(p_{H,2}) &= 0, \quad \forall\, p_{H,2}, \end{aligned} \tag{1.6}$$

$$V_1(p_{H,2}) = p_{H,2}\theta_H\left[1 - \frac{U_{H,1}(p_{H,2})}{\Delta\theta}\right] + \frac{U_{H,1}(p_{H,2})}{\Delta\theta}\theta_L. \tag{1.7}$$

As the continuation values for the low-type are zero for every prior, his payoffs at $r = 2$ are only his instant payoff, while the payoffs for the high-type buyer are the sum of the instant payoffs and his continuation value at (1.6).

Next, we solve the seller's problem at (1.5) for $r = 2$ after including in it the simplifications of Section 2.1 To do that we propose in the following lemma a reduced program equivalent to the seller's problem.

[13] We could choose any other value for α_1 and the main result of the paper will still hold. Bester and Strausz specification allows the possibility of giving to the seller the option of choosing α_1 at period $r = 2$. Using an example, it can be shown for $r = 2$ that the seller prefers $\alpha_1 = 1$ at next period when her prior is lower than $\frac{\theta_L}{\theta_H}$, and $\alpha_1 = 0$ when her prior is higher than $\frac{\theta_L}{\theta_H}$. Including this action for the seller complicates the model without upsetting our result.

Lemma 5 *The seller's problem at (1.5) for $r = 2$ is equivalent to the reduced program:*

$$\underset{\{q_2,\Gamma_2\}}{Max} \sum_{i=L,H} \sum_{m_2=l,h} p_{i,3} q_i(m_2) \left[v_2(m_2) + \delta V_1(p_{H,2}(m_2))\right], \quad \textit{subject to,} \tag{1.8}$$

$$IC^*_{H,2}: \quad u_{H,2}(h) + \delta U_{H,1}\left(p_2\left(h\right)\right) = u_{H,2}(l) + \delta U_{H,1}\left(p_2\left(l\right)\right),$$

$$IR^*_{L,2}: \quad u_{L,2}(l) + \delta U_{L,1}\left(p_2\left(l\right)\right) = 0,$$

$$SMC_2: x_2(h) - x_2(l) \geq \delta \frac{U_{H,1}\left(p_2(l)\right)}{\Delta\theta} - \delta \frac{U_{H,1}\left(p_2(h)\right)}{\Delta\theta}, \quad \textit{with equality if } q_L > 0,$$

$$BR_2: \quad p_{i,2}\left(m_2\right) = \frac{p_{i,3} q_i\left(m_2\right)}{\sum\limits_{k=L,H} p_{k,3} q_k(m_2)}, \; m_2 = l, h, \; i = L, H$$

$$x_2 \in [0,1], \; q_H > 0, \; q_L < 1.$$

Proof. See the Appendix ■

The previous lemma proves that (1.5), after simplifications of Section 2.1, is equivalent to (1.8). The intuitive description of the constraints in the latter problem is given as follows, and next we describe the intuition of the proof. The seller considers a binding incentive compatibility constraint for the high-type ($IC^*_{H,2}$), a binding individual rationality for the low-type ($IR^*_{L,2}$) and a new constraint, the *Sequential Monotonicity Condition for* $r = 2$ (SMC_2), which replaces the incentive compatibility of low-type.

The proof has the following steps. First, the incentive compatibility of the high-type ($IC_{H,2}$) jointly with the individual rationality of the low-type ($IR_{L,2}$) imply that individual rationality of the high-type ($IR_{H,2}$) is always satisfied. Second, notice that the seller can increase the payment of the low-type buyer and the one of the high-type buyer in the same amount while maintaining the incentive compatibility of high-type buyer and the individual rationality of low-type buyer. It is optimal for the seller to increase the payments until the low-type buyer's payment extracts all his surplus, resulting in $IR^*_{L,2}$. This payment is the maximum value that the low-type buyer can pay without retreating from the mechanism. Once the seller fixes the low-type buyer's payment, she continues increasing the high-type buyer's payment until the high-type buyer is indifferent to reporting the truth or not, i.e., $IC^*_{H,2}$. Because the high-type buyer is indifferent to both messages, the requirement that he must tell the truth with positive probability is satisfied. Finally, assuming $IC^*_{H,2}$ and $IR^*_{L,2}$, the incentive compatibility of the low-type buyer ($IC_{L,2}$) is equivalent to the SMC_2. This new restriction plays a similar role as the monotonicity condition of the static case, which asks to allocation to be increasing in the buyer type. In fact, notice that if $\delta = 0$, our model collapses to the static case and the SMC_2 to the monotonicity condition. In this dynamic framework, the SMC_2 is more restrictive. It still asks that the current allocation increases in the buyer type. It also requires that the difference in current allocations must be at least as large as the difference between the future discounted payoffs (weighted by $\Delta\theta$) that the high-type buyer gets by lying and the payoffs he gets by telling the truth.

Operating with $IC^*_{H,2}$ and $IR^*_{L,2}$ and plugging them into the seller's objective function, the seller's problem at (1.8) becomes

$$\underset{\{q_2,x_2\}}{Max}\ x_2(l)\theta_L + \rho_{H,3}\left(x_2(h) - x_2(l)\right)\theta_H + \delta\rho_{H,3}\left[U_{H,1}\left(p_{H,2}(h)\right) - U_{H,1}\left(p_{H,2}(l)\right)\right] \qquad (1.9)$$
$$+\,\delta\left[\rho_{H,3}V_1(p_{H,2}(h)) + \left(1-\rho_{H,3}\right)V_1(p_{H,2}(l))\right],$$

subject to,

$SMC_2,\ BR_2,\ x_2 \in [0,1],\ q_H > 0,\ and\ q_L < 1.$

where $\rho_{H,3} = (p_{H,3}q_H + p_{L,3}q_L)$ is the total probability of observing a message h.

To solve the game it is useful to introduce the following definition.

Definition 6 *A mechanism induces* learning *when* $U_{H,1}\left(p_{H,2}(l)\right) - U_{H,1}\left(p_{H,2}(h)\right) \neq 0$.

When $U_{H,1}\left(p_{H,2}(l)\right) - U_{H,1}\left(p_{H,2}(h)\right) = 0$, at $r = 1$ either $p_{H,2}(h) \geq p_{H,2}(l) \geq \frac{\theta_L}{\theta_H}$ or $\frac{\theta_L}{\theta_H} > p_{H,2}(h) \geq p_{H,2}(l)$ from (1.6). From the optimal solution for the last period, in the former case the seller will propose a price posting equal to θ_H and θ_L in the latter case. Hence, her expected continuation value is linear on $p_{H,3}$ and equal to $V_1\left(p_{H,3}\right)$.[14] On the other hand, when $U_{H,1}\left(p_{H,2}(l)\right) - U_{H,1}\left(p_{H,2}(h)\right) \neq 0$, it must be that $p_{H,2}(h) \geq \frac{\theta_L}{\theta_H} > p_{H,2}(l)$ from (1.6) and $q_H \geq q_L$, giving $U_{H,1}\left(p_{H,2}(l)\right) - U_{H,1}\left(p_{H,2}(h)\right) = \Delta\theta$. Therefore, the price posting to be proposed at $r = 1$ differs depending on the message observed in the first period: θ_H in case of observing message h and θ_L in case of l. The interpretation indicates that learning becomes relevant when it induces the seller to offer a different mechanism in the future for each message today, changing the buyer's payoffs.

To solve the problem, we split it into two subproblems. First, we take q_2 as given and we solve with respect to x_2. Second, we solve with respect to q_2 using the allocations we obtained in the first step.[15] So, the first step problem is given by

$$\underset{\{x_2\}}{Max}\ x_2(l)\,\theta_L + \rho_{H,3}\left\{\left[x_2(h) - x_2(l)\right]\theta_H\right\} + \delta A,\quad \text{subject to,} \qquad (1.10)$$
$$SMC_2 : x_2(h) - x_2(l) \geq \delta\frac{U_{H,1}\left(p_{H,2}(l)\right)}{\Delta\theta} - \delta\frac{U_{H,1}\left(p_{H,2}(h)\right)}{\Delta\theta},\quad with\ equality\ if\ q_L > 0,$$
$$x_2 \in [0,1],$$

where

$$A = \rho_{H,3}\left[U_{H,1}\left(p_{H,2}(h)\right) - U_{H,1}\left(p_{H,2}(l)\right)\right] + \rho_{H,3}V_1(p_{H,2}(h)) + \left(1-\rho_{H,3}\right)V_1(p_{H,2}(l)).$$

A is a constant that has all those terms in the objective function at (1.10) no depending on x_2.

Note that because the seller's payoffs are increasing in $x_2(h)$ and that an increment of $x_2(h)$ relaxes the SMC_2, then the optimal $x_2(h)$ is 1. On the other hand, to obtain the allocation for message l, we can differentiate the two situations: when $q_L = 0$ and when $q_L \neq 0$. In the former,

[14] In other words, the convex combination of seller's continuation values, $\rho_{H,3}V_1(p_{H,2}(h)) + \left(1-\rho_{H,3}\right)V_1(p_{H,2}(l))$ is equal to $V_1(p_{H,3})$. There are two possible situations when there is no learning: when $p_{H,3} < \frac{\theta_L}{\theta_H}$, where the seller's continuation values are $V_1(p_{H,2}(h)) = V_1(p_{H,2}(l)) = \theta_L$, which are equal to $V_1(p_{H,3}) = \theta_L$; and when $p_{H,3} \geq \frac{\theta_L}{\theta_H}$, where $V_1(p_{H,2}(h)) = p_{H,2}(h)\theta_H$ and $V_1(p_{H,2}(l)) = p_{H,2}(l)\theta_H$, equal to $V_1(p_{H,3}) = p_{H,3}\theta_H$.

[15] We are using the general property $\underset{\{x,y\}}{Max} f(x,y) = \underset{\{x\}}{Max}\left(\underset{\{y\}}{Max} f(x,y)\right)$.

$x_2(l)$ depends on $\rho_{H,3} = p_{H,3}q_H + (1 - p_{H,3})q_L$ and is given by

$$x_2(l) = \begin{cases} 0 & if \quad \rho_{H,3} > \frac{\theta_L}{\theta_H}, \\ \alpha_2 & if \quad \rho_{H,3} = \frac{\theta_L}{\theta_H}, \\ \mu & if \quad \rho_{H,3} < \frac{\theta_L}{\theta_H}, \end{cases}, \tag{1.11}$$

with $\alpha_2 \in [0, \mu]$ and $\mu = \min\left\{1, x_2(h) - \delta\frac{U_{H,1}(p_{H,2}(l))}{\Delta\theta} + \delta\frac{U_{H,1}(p_{H,2}(h))}{\Delta\theta}\right\}$. When $\rho_{H,3} = \frac{\theta_L}{\theta_H}$, the seller's payoffs are constant for any $x_2(l) \in [0, \mu]$. Then, to simplify the analysis we assume $x_2(l) = 0$ at $\rho_{H,3} = \frac{\theta_L}{\theta_H}$.

When $q_L \neq 0$, the low-type is indifferent to both messages and the SMC_2 holds with equality, restricting the value of $x_2(l)$, which is now given by

$$x_2(l) = x_2(h) - \delta\frac{U_{H,1}(p_{H,2}(l))}{\Delta\theta} + \delta\frac{U_{H,1}(p_{H,2}(h))}{\Delta\theta}. \tag{1.12}$$

To solve the second subproblem, we differentiate those cases where $x_2(l) = 0$ and where $x_2(l) \neq 0$.

Definition 7 *We say that a mechanism has* SMC non-binding *if* $x_2(l) = 0$ *and* SMC binding *if* $x_2(l) \neq 0$.

In both cases, it is possible to have *learning* or *no-learning*. Because $x_2(h) = 1$, $x_2(l) = 0$ occurs only when $q_L = 0$ and $\rho_{H,3} \geq \frac{\theta_L}{\theta_H}$ from (1.11). On the other hand, $x_2(l) \neq 0$ occurs either when $q_L = 0$ and $\rho_{H,3} < \frac{\theta_L}{\theta_H}$, or when $q_L \neq 0$. In both cases, by (1.11) or (1.12), respectively, $x_2(l) = 1 - \delta$ when there is *learning* and $x_2(l) = 1$ when there is *no-learning*.

1. *SMC binding with no-learning* (SMC*+NL)
   By *no-learning*, $U_{H,1}(p_{H,2}(l)) - U_{H,1}(p_{H,2}(h)) = 0$. Then, the expected continuation value for the seller is equal to $V_1(p_{H,3})$ (see above). By *SMC binding*, $x_2(l) \neq 0$. From (1.12) and *no-learning* it must be that $x_2(l) = 1$. Substituting continuation values and allocations at (1.10) and after some simplifications results in the seller's maximum payoffs equal to

   $$\theta_L + \delta \max\{p_{H,3}\theta_H, \theta_L\}.$$

   The seller is indifferent among any pair (q_L, q_H) such that there is *SMC binding with no-learning*. We can assume $q_L = q_H \neq 0$ (i.e. $p_{H,2}(h) = p_{H,2}(l) = p_{H,3}$) without a loss of generality.

2. *SMC binding with learning* (SMC*+L)
   Now $U_{H,1}(p_{H,2}(l)) - U_{H,1}(p_{H,2}(h)) \neq 0$ by *learning*. Because $p_{H,2}(h) \geq \frac{\theta_L}{\theta_H} > p_{H,2}(l)$, $U_{H,1}(p_{H,2}(l)) - U_{H,1}(p_{H,2}(h)) = \Delta\theta$ by (1.6) and, $V_1(p_{H,2}(h)) = p_{H,2}(h)\theta_H$ and $V_1(p_{H,2}(l)) = \theta_L$ by (1.7). By *SMC binding* $x_2(l) \neq 0$, and jointly with *learning*, $x_2(l) = 1 - \delta$. Substituting allocations and continuation values at (1.10) and after some simplifications, the seller chooses (q_L, q_H) to maximize

   $$\theta_L + \delta\rho_{H,3}p_{H,2}(h)\theta_H.$$

According to Bayes' rule $\rho_{H,3} p_{H,2}(h) = p_{H,3} q_H$. Because $q_H \leq 1$, a mechanism with *SMC binding with learning* is weakly dominated by a mechanism under case **1** for any prior.

3. *SMC non-binding with learning* (SMC+L)
Now $U_{H,1}(p_{H,2}(l)) - U_{H,1}(p_{H,2}(h)) = \Delta\theta$ by *learning* and $x_2(l) = 0$ by *SMC non-binding*. The allocation $x_2(l) = 0$ implies that $q_L = 0$ and $\rho_{H,3} \geq \frac{\theta_L}{\theta_H}$, i.e. $q_H \geq \frac{\theta_L}{\theta_H p_{H,3}}$. This last requirement jointly with $p_{H,2}(h) \geq \frac{\theta_L}{\theta_H} > p_{H,2}(l)$ (by *learning*) implies $p_{H,3} \geq \frac{\theta_L}{\theta_H}$ (see Lemma 4 in the Appendix). From (1.7), $V_1(p_{H,2}(h)) = p_{H,2}(h)\,\theta_H$ and $V_1(p_{H,2}(l)) = \theta_L$. Therefore, the seller chooses q_H to maximize

$$\rho_{H,3}\theta_H + \delta\theta_L,$$
$$\text{subject to } q_L = 0,\ \rho_{H,3} \geq \frac{\theta_L}{\theta_H},\ p_{H,2}(l) < \frac{\theta_L}{\theta_H},$$

getting

$$p_{H,3}\theta_H + \delta\theta_L.$$

when $q_H = 1$.

4. *SMC non-binding with no-learning* (SMC+NL)
no-learning means $U_{H,1}(p_{H,2}(l)) - U_{H,1}(p_{H,2}(h)) = 0$ with expected continuation value for the seller equals to $V_1(p_{H,3})$ (see above). *SMC non-binding* means $x_2(l) = 0$. The necessary conditions for the optimum of the first problem implies that $q_L = 0$ and $\rho_{H,3} \geq \frac{\theta_L}{\theta_H}$, i.e. $q_H \geq \frac{\theta_L}{\theta_H p_{H,3}}$. Additionally, by *no-learning*, it must be that $p_{H,2}(l) \geq \frac{\theta_L}{\theta_H}$.[16] To satisfy previous requirements it is necessary that $p_{H,3} \geq p^*$ (where $p^* = \frac{\theta_L}{\theta_H^2}\Delta\theta + \frac{\theta_L}{\theta_H}$, see Lemma 4 in the Appendix). By substitution at (1.10) and simplification, we see that the seller chooses q_H to maximize

$$\rho_{H,3}\theta_H + \delta p_{H,3}\theta_H,$$
$$\text{subject to } q_L = 0,\ \rho_{H,3} \geq \frac{\theta_L}{\theta_H},\ p_{H,2}(l) \geq \frac{\theta_L}{\theta_H},$$

getting

$$\frac{p_{H,3}\theta_H - \theta_L}{\Delta\theta}\theta_H + \delta p_{H,3}\theta_H,$$

with $q_H = \frac{p_{H,3}\theta_H - \theta_L}{p_{H,3}\Delta\theta}$.

[16] $q_L = 0$ gives $p_{H,2}(h) = 1$ by BR_2. Then, by *non-learning*, it must be that $p_{H,2}(h) \geq p_{H,2}(l) \geq \frac{\theta_L}{\theta_H}$.

Notice that at a prior $p_{H,3} = \frac{\theta_L}{\theta_H}\left(\frac{\theta_H+\delta\Delta\theta}{\theta_L+\delta\Delta\theta}\right)$, the seller is indifferent to a mechanism of case SMC+L and a mechanism of case SMC+NL. From now on, we refer to this prior as τ_2 and we assume, without a loss of generality, that when $p_{H,3} = \tau_2$ the seller proposes a mechanism of case SMC+NL. At a prior equal to $\frac{\theta_L}{\theta_H}$, case SMC*+NL, case SMC*+L and case SMC+L give the same payoffs to the seller. We assume without a loss of generality that the seller proposes a mechanism of case SMC+L at that prior.

The following figure summarizes maximum the seller's payoffs from each of the previous options for every prior in those ranges in which they are defined. Case SMC*+L is dominated by case SMC*+NL (they are coincident when $p_{H,3} \geq \frac{\theta_L}{\theta_H}$). Case SMC*+NL is piecewise linear in p, but the range with $p_{H,3} \geq \frac{\theta_L}{\theta_H}$ is dominated by cases SMC+L and SMC+NL. These last cases are also linear in p. Case SMC+L has a lower slope than case SMC+NL and they cross each other at τ_2.

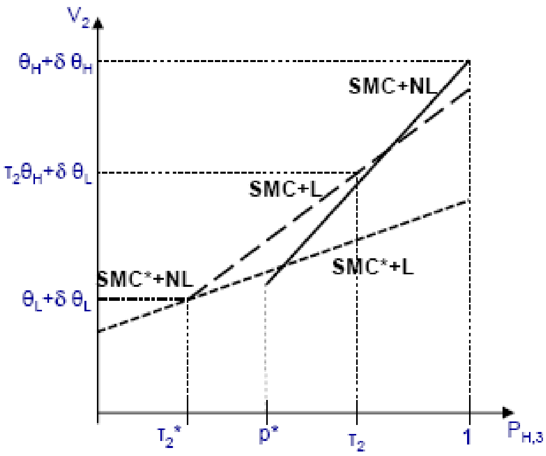

Figure 1: Seller's payoff under different priors and mechanisms. In Dot-line case **SMC*+NL**; in Small-dash-line case **SMC*+L** (coincident with case **SMC*+NL** when $p_{H,3} \geq \tau_2^*$); in Dash-line case **SMC+L;** in Solid-line case **SMC+NL**

As in the last period, at $r = 2$ a prior equal to $\frac{\theta_L}{\theta_H}$ acts as a threshold. Therefore, when $p_{H,3} \geq \frac{\theta_L}{\theta_H}$ the seller finds it optimal to implement one of the mechanisms with *SMC non-binding* and, when $p_{H,3} < \frac{\theta_L}{\theta_H}$, the *SMC binding with no-learning*. We denote this threshold with τ_2^*. It is straightforward to check under which prior the seller finds optimal to choose each mechanism. This is stated in next proposition.

Proposition 8 *The optimal selling mechanism verifies that:*

- *if* $p_{H,3} < \tau_2^*$, (SMC binding with no-learning) *with* $x_2(h) = 1$, $x_2(l) = 1$, $w_2(h) = \theta_L$, $w_2(l) = \theta_L$, *and* $q_H = q_L \neq 0$.

- *if* $p_{H,3} \in [\tau_2^*, \tau_2)$, (SMC non-binding with learning) *with* $x_2(h) = 1$, $x_2(l) = 0$, $w_2(h) = \theta_H - \delta\Delta\theta$, $w_2(l) = 0$, $q_H = 1$ *and* $q_L = 0$.

- *if* $p_{H,3} \geq \tau_2$, (SMC non-binding with no-learning) *with* $x_2(h) = 1$, $x_2(l) = 0$, $w_2(h) = \theta_H$, $w_2(l) = 0$, $q_H = \frac{p_{H,3}\theta_H - \theta_L}{p_{H,3}\Delta\theta}$ *and* $q_L = 0$.

Proof. Directly from previous analysis. It only remains to get $w_2(h)$ and $w_2(l)$. As these variables are a mechanic substitution in $IC^*_{H,2}$ and $IR^*_{L,2}$, they are relegated to the Appendix. ■

We now describe the thought behind the results of the proposition. The first case corresponds to the case in which the seller is *pessimistic*, i.e., $p_{H,3} < \frac{\theta_L}{\theta_H}$. In this case she always sells at a price equal to the low-type value. This corresponds to case SMC*+NL. The seller has only one alternative, mentioned above in SMC*+L, under which the seller sells with probability 1 to a high-type buyer and with probability $1-\delta$ to a low-type buyer. Learning means that the optimal mechanism in the next period after a message h only sells to the high-type buyer and at a price equal this buyer's value, and after a message l sells to both types at a price equal to the low-type value. Thus, *learning* reduces the expected payoffs of the first period in $\delta\theta_L$ and increases the payoffs of the second period in $\delta p_{H,3}\theta_H$, and hence it is not optimal.

When the seller is *optimistic*, i.e., $p_{H,3} \geq \frac{\theta_L}{\theta_H}$, a mechanism from case SMC+L or from case SMC+NL is the optimal one. In both cases the seller offers a mechanism such that, in case of observing a message h, the optimal mechanism in the next period is to sell only to a high-type buyer at a price equal to this buyer's value. In particular, if $p_{H,3} \geq \tau_2$ (the seller is *extremely optimistic*), she prefers a mechanism from case SMC+NL over a mechanism from case SMC+L. In such a case, there is *no-learning* and the optimal mechanism is such that she sells only to the high-type buyer in the second period no matter the message observed in the first period.[17] Therefore, in the second period, the buyer always makes zero surplus. On the other hand, when the seller is *moderately optimistic, i.e.,* $p_{H,3} \in \left[\frac{\theta_L}{\theta_H}, \tau_2\right)$, the optimal mechanism is with *learning*. In the next period, the seller sells only to a high-type buyer at a price equal to this buyer's value in the case of observing h, or sells to both types at a price equal to the low-type value in the case of observing l. The seller has to pay a bribe to incentive the high-type buyer to reveal his type. This bribe is equal to his discounted future losses by being discriminated in the second period. Therefore, the buyer makes zero surplus in the second period in case the of reporting h or positive surplus in case of reporting l.[18] Alternatively, using a mechanism of case SMC*+NL (or SMC*+L), the seller can obtain the same posteriors (and as consequence the same continuation values) as in case SMC+NL (or SMC+L) but, since she has to keep both buyer types indifferent between messages, she has to ask for a lower payment in the first period.

Finally, notice that the seller becomes optimistic very fast when a high-type message is sent in a mechanism with *SMC non-binding*, reaching $p_{H,2}(h) = 1$ in both situations. On the other hand, the seller slowly becomes pessimistic when she observes a low-type message (this last effect can be observed more clearly with more periods in the second chapter).

Figure 2 summarizes optimal belief dynamic for each prior.

[17] The seller picks a q_H that "commits" her to sell in the second period to the high-type buyer at a price equal to this buyer's value while asking a high payment in the first period. This q_H is lower than one (assigning positive probability of lying to a high-type), keeping her optimistic enough in the case of observing a message l.

[18] To offer this bribe, the seller considers that the high-type is going to report the truth with probability of one, i.e., she picks $q_H = 1$.

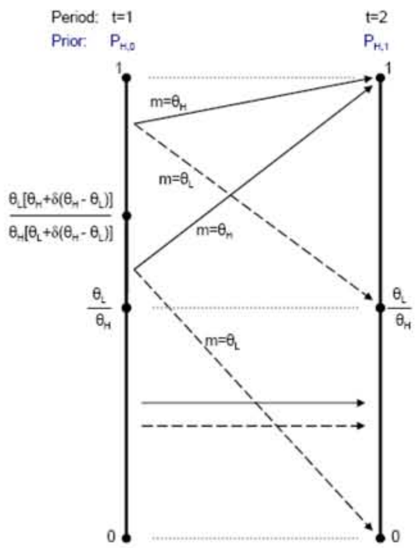

Figure 2: Optimal belief dynamic under different priors. Vertical line at the left represents the prior at $r = 2$. Vertical line at the right represents the posteriors. Full-line arrow is the belief dynamic under a message h. Dot-line arrow is the belief dynamic under a message l.

Proposition 1 gave us a characterization of the optimal selling mechanisms for any prior. We state in the following corollary that these optimal mechanisms, which are direct mechanisms, can be implemented by price posting. To do so, we propose an alternative outcome $(\hat{q}_2, \hat{p}_2, \hat{\Gamma}_2)$ where $\hat{\Gamma}_2$ is a price posting mechanism. Next, we check for any prior whether this outcome is payoff equivalent to the incentive efficient outcome (q_2, p_2, Γ_2) that solves (1.8) and contains the optimal selling mechanism characterized in previous propositions. Because the proof is mechanic, we relegate it to the Appendix.

Corollary 9 *The optimal selling mechanism at $r = 2$ can be implemented by a price posting. In particular:*
1) when $p_{H,3} \geq \tau_2$, the price is θ_H, the high-type buyer randomizes with probability $\frac{p_{H,3}\theta_H - \theta_L}{p_{H,3}\Delta\theta}$ and low-type buyer never buys,
2) when $p_{H,3} \in \left[\frac{\theta_L}{\theta_H}, \tau_2\right)$, the price is $\theta_H - \delta\Delta\theta$, the high-type buyer always buys and low-type buyer never buys, and
3) when $p_{H,3} < \frac{\theta_L}{\theta_H}$, the price is θ_L, both types always buy.

Proof. See the Appendix ■

These results reassert that the price posting mechanisms proposed by Bolton and Dewatripont (2005) are the optimal mechanisms when we are not restricted to take-it-or-leave-it offers. When $p_{H,3} < \frac{\theta_L}{\theta_H}$, the optimal mechanism is a price posting with *SMC binding with no-learning.* The authors call this mechanism "pooling". They call "separation" to the optimal price posting mechanism (the *SMC non-binding with learning*) when $p_{H,3} \in \left[\frac{\theta_L}{\theta_H}, \tau_2\right)$ and "semi-separation" to the optimal price posting mechanism (*SMC non-binding with no-learning*) when $p_{H,3} \geq \tau_2$.

1.4 Difference in patience

In this section we show that when buyer and seller differ in their patience price posting is no longer optimal.

Suppose that both players have different discount factors: β for the seller, δ for the buyer. When $\beta = \delta$ we are in the case analyzed in previous sections, i.e. price posting is optimal. When $\beta \neq \delta$, we can use the same arguments of previous section to deduce that seller's problem is now

$$\underset{\{q_2,x_2\}}{Max}\ x_2(l)\theta_L + \rho_{H,3}\left(x_2(h) - x_2(l)\right)\theta_H + \delta\rho_{H,3}\left[U_{H,1}\left(p_{H,2}(h)\right) - U_{H,1}\left(p_{H,2}(l)\right)\right]$$
$$+\beta\left[\rho_{H,3}V_1(p_{H,2}\left(h\right)) + \left(1-\rho_{H,3}\right)V_1(p_{H,2}\left(l\right))\right],$$

subject to,

$SMC_2,\ BR_2,\ x_2 \in [0,1],\ q_H > 0,\ and\ q_L < 1.$

Notice that this problem is the same that the one at (1.9) with the only change in the discount factor that affects seller's continuation values, where δ was replaced by β.

Using the same procedure than in Section 3, the seller gets the following maximum payoffs in each case:

1. *SMC binding with no-learning:* with $x_2\left(h\right) = 1$, $x_2\left(l\right) = 1$, $w_2\left(h\right) = \theta_L$, $w_2\left(l\right) = \theta_L$, and $q_H = q_L \neq 0$.

   $$\theta_L + \beta\max\left\{p_{H,3}\theta_H, \theta_L\right\}.$$

2. *SMC binding with learning:* with $x_2\left(h\right) = 1$, $x_2\left(l\right) = 1-\delta$, $w_2\left(h\right) = \theta_L$, $w_2\left(l\right) = (1-\delta)\theta_L$, and $q_H = 1$, $q_L = 0$,

   $$\theta_L + \left(\beta - \delta\right)\left(1 - p_{H,3}\right)\theta_L + \beta p_{H,3}\theta_H.$$

3. *SMC non-binding with learning:* (defined for $p_{H,3} \geq \frac{\theta_L}{\theta_H}$) with $x_2\left(h\right) = 1$, $x_2\left(l\right) = 0$, $w_2\left(h\right) = \theta_H - \delta\Delta\theta$, $w_2\left(l\right) = 0$, and $q_H = 1$, $q_L = 0$,

   $$p_{H,3}\theta_H + \left(\beta - \delta\right)p_{H,3}\Delta\theta + \beta\theta_L.$$

4. *SMC non-binding with no-learning:* (defined for $p_{H,3} \geq p^*$) with $x_2\left(h\right) = 1$, $x_2\left(l\right) = 0$, $w_2\left(h\right) = \theta_H$, $w_2\left(l\right) = 0$, and $q_H = \frac{p_{H,3}\theta_H - \theta_L}{p_{H,3}\Delta\theta}$, $q_L = 0$,

   $$\frac{p_{H,3}\theta_H - \theta_L}{\Delta\theta}\theta_H + \beta p_{H,3}\theta_H.$$

Figure 3 represents seller's payoffs under different mechanisms when $\delta = \beta$ (at the left) and for a generic case of $\delta < \beta$ (at the right). In the later, for any difference in the discount factors it is possible to find a $p_{H,3} < \frac{\theta_L}{\theta_H}$ such that price posting is not optimal.

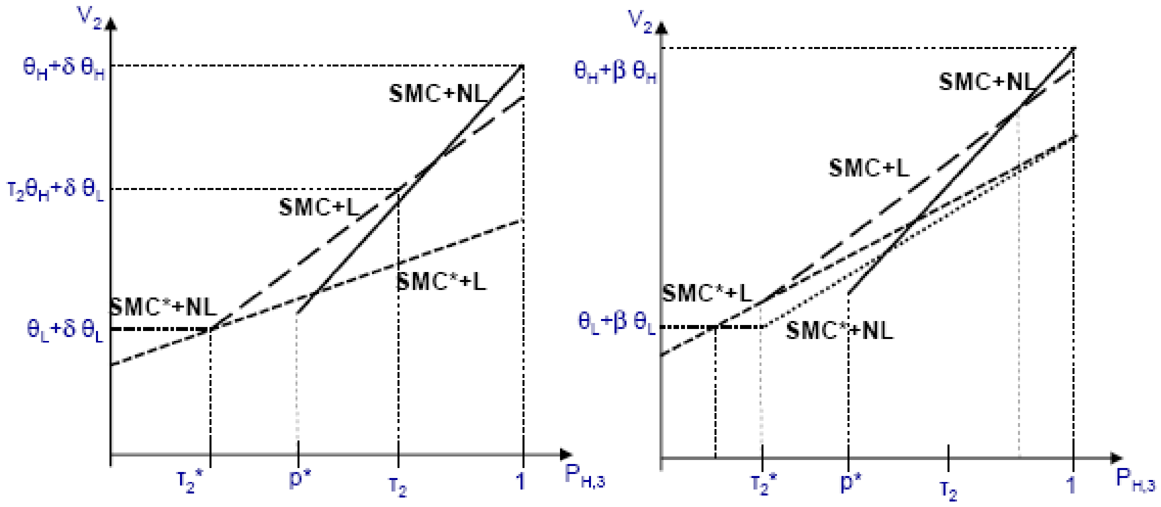

Figure 3: In Dot-line case **SMC*+NL**; in Small-dash-line case **SMC*+L** (coincident with case **SMC*+NL** when $p_{H,3} \geq \tau_2^*$ in the graph at the left); in Dash-line case **SMC+L;** in Solid-line case **SMC+NL**.

In the following example we illustrate, for a particular prior, how large has to be the difference between discount factors to have that price posting is not optimal.

Example 10 Different Discount Factors:

Let consider $\theta_L = 1$*,* $\theta_H = 2$*, a prior* $p_{H,3} = \frac{1}{3}$ *and* $\beta = 1$*. Case SMC+L and SMC+NL are not defined for* $p_{H,3} < \frac{\theta_L}{\theta_H}$*.*

Seller's payoffs using a mechanism from case SMC+NL are equal to* $\theta_L + \beta\theta_L$*, i.e.* $V_2 = 2.0$*. Seller's payoffs using a mechanism from case SMC*+L are equal to* $\theta_L + (\beta - \delta)(1 - p_{H,3})\theta_L + \beta p_{H,3}\theta_H$*, i.e.* $V_2 = 1 + (1-\delta)\frac{2}{3} + \frac{2}{3}$*. Choosing the appropriate value for* δ*, previous seller's payoffs can be larger than* $\theta_L + \beta\theta_L$*.*

The next chart shows how seller's payoffs change with δ*:*

Γ_2 / δ	**Case 1** $x_2(h)=x_2(l)=1$, $w_2(h)=w_2(l)=\theta_L$, $q_H=q_L\neq 0$, $V_2=$	**Case 2** $x_2(h)=1$, $x_2(l)=1-\delta$, $w_2(h)=\theta_L$, $w_2(l)=(1-\delta)\,\theta_L$, $q_H=1$, $q_L=0$, $V_2=$
1.00	2.0	1.67
0.75	2.0	1.83
0.50	2.0	2.0
0.25	2.0	2.17
0.00	2.0	---

When $\delta = 0.25$, a mechanism with SMC+L (case **2**) maximizes seller's payoffs making $V_2 = 2.17$. Buyer's payoffs are equal to 1 for the high-type buyer and 0 for the low-type buyer. This mechanism cannot be implemented by a price posting (see the Appendix).*

In previous example we have chosen a particular prior to illustrate our point. In the example the seller is moderately pessimistic about facing a high-type buyer. This is, she believes that the probably of facing a high-type buyer is small, but it is still large enough to leave room for finding optimal to learn when, at the same time, she is relatively more patient than the buyer. Notice that seller's payoffs under SMC*+L (case **2**) are larger than those under SMC*+NL (case **1**) when $\frac{\beta \Delta\theta p_{H,3}}{\theta_L(1-p_{H,3})} > \delta$. In the example, it occurs when $0.5 > \delta$.

1.4.1 Goethe's Mechanism

In this subsection we show that our model explains why the mechanism proposed by Goethe (see the Introduction) may be optimal when price posting is not.

To prove so, we take previous example, we construct a variation of the mechanism à la Goethe and its equilibrium, and we show that payoffs of this mechanism are arbitrarily close to payoffs of the optimal mechanism in Example 1.

Example 11 Goethe's Mechanism:

Publisher valuations are $\theta_L = 1$ or $\theta_H = 2$. Goethe has a prior $p_{H,3} = \frac{1}{3}$. Discount factors are $\beta = 1$ for Goethe and $\delta = 0.25$ for the publisher.

At last period $r = 1$, Goethe uses the optimal price posting mechanism described at the beginning of Section 3.

At $r = 2$, Goethe proposes to the publisher the following mechanism:

- Goethe sends to a lawyer a sealed envelope with his reservation price $R \in \Re^+$. Previously, Goethe commits with the publisher to the probability with which he will send each possible value of R.[19]
- At the same time, the publisher sends to the same lawyer a sealed envelope with his offer $m \in \Re^+$.
- If $m \geq R$, sale takes place at price R (i.e. $x(m) = 1, w(m) = R$). If $m < R$, the good is not sold (i.e. $x(m) = 0, w(m) = 0$).

An equilibrium for this mechanism is:

- Goethe commits to send a reservation price $R_1 = \theta_L + \varepsilon$ with probability p, and $R_2 = \theta_L$ with probability $(1 - p)$, where $p = \frac{\delta\Delta\theta}{\Delta\theta - \varepsilon}$.
- high-type reports $m_1 = \theta_L + \varepsilon$ and low-type reports $m_2 = \theta_L$.

[19] Notice that this is a variation of the mechanism proposed by Goethe described at the Introduction, where he does not commit to the probability with which he will send each reservation price. We assume this commitment of Goethe to construct the equilibrium below.

Publisher's payoffs for message m_1 and m_2 ($U_{i,2}(m_1)$ and $U_{i,2}(m_2)$, respectively, where $i \in \{L, H\}$) are:

$$\begin{aligned}
U_{H,2}(m_1) &= p(\Delta\theta + \varepsilon) + (1-p)\Delta\theta, \\
U_{H,2}(m_2) &= p\delta\Delta\theta + (1-p)(\Delta\theta + \delta\Delta\theta), \\
U_{L,2}(m_1) &= -p\varepsilon, \\
U_{L,2}(m_2) &= 0.
\end{aligned}$$

Notice that $U_{H,2}(m_1) > U_{H,2}(m_2)$ and $U_{L,2}(m_2) > U_{L,2}(m_1)$ when $\varepsilon > 0$. It follows that each type reports his respective message with probability one revealing their types. High-type buyer gets the poem no matter the reservation price sent by Goethe and low-type buyer gets it only in case of R_2. Goethe's payoffs are

$$V_2 = p_{H,3}\left[pR_1 + (1-p)R_2 + \beta\theta_H\right] + (1-p_{H,3})\left[(1-p)R_2 + \beta\theta_L\right].$$

When $\varepsilon \to 0$, types are almost indifferent between messages with $U_{H,2}(\cdot) \to \Delta\theta$ and $U_{L,2}(\cdot) \to 0$ and Goethe makes $V_2 \to 2.17$. We showed in Example 1 that the optimal mechanism gives $V_2 = 2.17$, then Goethe's Mechanism is optimal in the limit.

1.5 Concluding Remarks

This paper establishes that the optimal selling mechanism when an uniformed seller with imperfect commitment faces the same consumer in a two periods game is to post a price in each one. This result holds whenever the difference in discount factors is small. Otherwise price posting is not optimal and the Goethe's Mechanism can be rationalized. The method used to solve this problem relies on the procedure propose by Bester and Strausz (2001).

In the related literature it is assumed that the seller uses a price posting as the selling mechanism. In this paper we find that there is not another mechanism that can be used by the seller to increase her profits. This is a limitation for the seller. She cannot propose a more complex mechanism -sacrificing payoff today in order to learn- to take advantage in the future. We also give some conditions under which our result does not hold.

This paper can be extended in many directions. The more natural extension is to generalize the model for more periods. This is the purpose of Chapter 2. Another interesting extension is increasing the number of types, when the monotone hazard rate property does not necessarily hold anymore.

It can be also analyzed the case with many buyers. Bester and Strausz (2000) shows that a direct mechanism with truthful reporting is not possible in a multi-buyer case. In the same direction, Evans and Reiche (2008) proves that the revelation principle fails in the multi-buyer setting but only if at least two buyers have private information. This last case can be considered to check the robustness of our result. To study an environment with more than one privately informed buyer we have to consider another approach.

Chapter 2

Optimal Selling Mechanism in a Repeated Game under Imperfect Commitment: The Multi-Period Case

2.1 Introduction

In Chapter 1 we proved that, in a two-period game, price posting is optimal when both players have the same discount factor but not when they are sufficiently different. In this paper, we extend the model of Chapter 1 to a finite number of periods larger than two when the discount factor is arbitrarily large and equal for both players.

Intuitively, allowing more than two periods provides a richer environment because the seller can now engage in a strategy of gradual learning. More formally, continuation values at any moment in time of a multi-period game may be a non-linear function in the prior for either the buyer or the seller. Moreover, in a static framework, price posting is an optimal mechanism when value functions are linear and it is not when they are not linear.[1] However, linearity (or piecewise linearity) in the prior on value functions is not a sufficient condition for optimality of price posting in an dynamic framework as we saw in Chapter 1. Then, it is reasonable to conjecture that price posting might not be optimal in a multi-period game.

We prove two things. First, the seller cannot do better than posting a price in every period as the selling mechanism without loss of generality. Second, in general along the equilibrium path the seller posts a price equal to the minimum buyer willingness to pay, i.e. the maximum competitive price. Discrimination between types is optimal only when the seller is extremely optimistic about facing a high-type consumer. In other case, learning, albeit possible, is so costly for the seller that it is not optimal. When the seller has the possibility of learning, her profits are reduced due to the strategic behavior of the buyer. We also give a complete characterization of the optimal mechanism and equilibrium payoffs for every prior.

As in Coase's conjecture, the monopolist cannot use a price above the competitive one to discrim-

[1] See Chapter 2 in Börgers (mimeo) or Chapter 2 in Bolton and Dewatripont (2005).

inate among buyers. Coase (1972) conjectured that a monopolist uses this price from the beginning when she has a durable good to sell in a finite numbers of periods. A solution to this conjecture is renting the durable good. This result implicitly assumes that the monopolist cannot track past buyer's decisions. However, in our framework the monopolist cannot commit to ignore the information disclosed by the buyer. Our model implies that there is no mechanism that solves the Coase's conjecture as a consequence of this lack of commitment.

This has also implications regarding the ratchet effect. In an arbitrary long game and if a discount factor is not too small, a privately informed buyer knows that in case of revealing his valuation in the current period he will not get any information rent thereafter (the ratchet effect). Then, the seller cannot induce him to reveal his information. Schmidt (1993) shows the presence of the ratchet effect on a repeated bargaining model, producing much pooling in all the equilibria of the game. In his work, the buyer (who has the bargaining power in his model) offers a price to a seller. As soon as a price higher than her production cost is accepted (revealing her type) the buyer will not give her any additional rents. This is true even if the price offered by the buyer in the current period is not the optimal one for him. Learning process, when it occurs, is always extreme. In our model, the seller (who has the bargaining power) can offer a more complex selling mechanism than price posting. For example, the seller can propose a menu of contracts such that if the high-type buyer buys the good in the current period, he is not completely revealing his valuation. In other words, in the following period the seller will not be certain about facing a high-type buyer. Therefore, she has to give him rents again if she wants to continue with her learning process. In contrast with Schmidt (1993), the seller can now propose mechanisms that allow her to learn gradually. Since we prove that these mechanisms are suboptimal, the seller cannot break the ratchet effect in equilibrium.

Skreta (2005) shows that her results at Skreta (2006) hold for the multi-period case. As we menttioned in Chapter 1, she studies a different framework: she considers the durable good case.

To solve the model we use a dynamic mechanism design approach following the procedure proposed in Bester and Strausz (2001), which provides a modified version of the revelation principle when there is imperfect commitment.

The rest of the paper is organized as follows. Section 2 provides a general set up of the problem and reviews the Bester and Strausz (2001) revelation principle for this kind of environment. Section 3 analyzes the problem with two types for any finite T periods game. Finally, Section 4 concludes. Those proofs considered relevant for the general understanding of the model are included in the main text while the rest can be found in the Appendix.

2.2 General Setup

Next, we propose a dynamic problem that follows the framework proposed by Bester and Strausz (2001) which is solved by recursive methods as they suggest. Therefore, in this section we directly propose a dynamic problem as a sequence of static problems. We show in the Appendix how our recursive formulation corresponds to the sequential problem.

Let's consider a multi-period game with $r = \{1, 2, ...T\}$ and $T < \infty$, where r is the number of periods remaining at the beginning of the current period. There is one risk neutral seller (the principal) and one risk neutral buyer (the agent) facing each other repeatedly. Both players discount the future at the same rate $\delta \in (0, 1]$. At every period, the seller can produce at zero cost a non-storable object that puts for sale to the buyer.[2] This buyer has valuation θ_i for the good, where $\theta_i \in \Theta = \{\theta_L, \theta_H\}$. We call θ_L (θ_H) the low-type buyer (high-type buyer) and sometimes we denote it by the subscript L (H). This valuation remains constant over time and is his private information. The initial probability of facing a high-type buyer is denoted by $p_{H,T+1}$, and for a low-type buyer by $p_{L,T+1} = 1 - p_{H,T+1}$. We refer to this as the prior of the seller.

A mechanism Γ_r in period r specifies a message set M_r and a decision function $y_r = (x_r, w_r)$, where $x_r : M_r \to [0,1]$ is the allocation rule and $w_r : M_r \to \mathbb{R}$ is the payment rule. Then, each element $m_r \in M_r$ commits the seller to implement the allocation rule $x_r(m_r)$ and requires for the buyer the payment $w_r(m_r)$.

The seller has imperfect commitment. This is, at every period the seller can commit herself to a mechanism for the current period but not for future ones. So, at the beginning of period r the seller chooses a mechanism $\Gamma_r \in \Upsilon$ given her prior $p_{H,r+1}$ about facing a high-type buyer, where Υ is the space of mechanisms. Next, the buyer observes Γ_r. His strategy specifies the probability $q_i(m_r)$ with which the agent sends each message m_r, where $q_i : M_r \to [0,1]$, for $i \in \{L, H\}$ and that verifies $\sum_{m_r \in M_r} q_i(m_r) = 1$. The buyer can always choose not to participate in the mechanism Γ_r.[3] In this case he gets zero instant payoffs but he can accept future ones. Next, the seller observes m_r and updates her beliefs about facing a high-type buyer. We denote it by $p_{H,r}(m_r)$ and is updated following a mapping $p_{H,r} : M_r \to [0,1]$. Beliefs constitute the state variable for the next period, i.e., $r-1$. In the following, we use $p_{L,r}(m_r)$ to indicate $1 - p_{H,r}(m_r)$ and $p_r(m_r)$ to indicate the vector of posteriors $(p_{L,r}(m_r), p_{H,r}(m_r))$ when a message m_r is sent.

We denote by $v_r(m_r)$ and $u_{i,r}(m_r)$ to the seller's and buyer's *instant* payoff, respectively, when the buyer with valuation θ_i sends the message m_r, i.e.

$$\begin{aligned} v_r(m_r) &= w_r(m_r), \\ u_{i,r}(m_r) &= x_r(m_r)\theta_i - w_r(m_r), \end{aligned}$$

$V_{r-1} : [0,1]^2 \to \mathbb{R}$ and $U_{i,r-1} : [0,1]^2 \to \mathbb{R}$ represent the continuation values for each player.[4]

Consequently, given the vector of priors $p_{r+1} \equiv (p_{L,r+1}, p_{H,r+1})$, the seller's problem at period r is to choose (q_r, p_r, Γ_r) that maximizes:

$$\sum_{i \in \Theta} \sum_{m_r \in M_r} p_{i,r+1} q_i\left(m_r\right)\left(v_r(m_r) + \delta V_{r-1}\left(p_r(m_r)\right)\right), \tag{2.1}$$

where $q_r \equiv \left(q_r\left(m_r\right)\right)_{m_r \in M_r}$ ($q_r\left(m_r\right)$ indicates the vector $(q_L(m_r), q_H(m_r))$), and $p_r \equiv \left(p_r\left(m_r\right)\right)_{m_r \in M_r}$, is subject to the following constraints:

[2] All our results hold for any constant production cost strictly less than the minimum possible willingness to pay of the buyer.

[3] Note that our definition of the mechanism requires participation. We take the usual convention that the buyer can decide whether to participate or not, getting zero payoffs in the last case. This convention is discussed later, when we talk about the individual rationality constraint (IR). Alternatively, it is possible to include a message in M_r that represents no participation.

[4] Continuation values depends on the vector of priors at the beginning of the period. Since there are two types, the vector of priors is completely determined by the prior about facing a high-type buyer, i.e. $p_{H,r+1}$. Then, later in the paper, and with some abuse of notation, continuation values will be represented as depending only in that prior, which we will denoted as p. We also will denote $p(m_r)$ to its posterior after observing m_r.

- The buyer's Incentive Compatibility ($IC_{i,r}$): the buyer chooses his optimal reporting strategy, i.e.,

$$\sum_{m_r \in M_r} q_i\left(m_r\right)\left(u_{i,r}(m_r) + \delta U_{i,r-1}\left(p_r\left(m_r\right)\right)\right) \geq \sum_{m_r \in M_r} q_i'\left(m_r\right)\left(u_{i,r}(m_r) + \delta U_{i,r-1}\left(p_r\left(m_r\right)\right)\right) \tag{2.2}$$

for $i \in \{L, H\}$, and for all $q_i'\left(m_r\right)$.

- The buyer's Individual Rationality ($IR_{i,r}$): The buyer's individual rationality constraint has to be satisfied for all types to which the seller assigns positive probability:

$$p_{i,r+1}\left[\sum_{m_r \in M_r} q_i\left(m_r\right)\left(u_{i,r}(m_r) + \delta U_{i,r-1}\left(p_r\left(m_r\right)\right)\right) - \delta \bar{U}_{i,r-1}\right] \geq 0 \tag{2.3}$$

for $i \in \{L, H\}$, where $\bar{U}_{i,r-1}$ is the continuation value when the buyer choose not to participate in the mechanism Γ_r. Although there is no loss of generality in assuming that the buyer participates with probability one, we have to warranty that he does not do better staying out. This implies $\bar{U}_{i,r-1} \geq 0$. We assume $\bar{U}_{i,1} = 0$ since it is the less restrictive in (2.3) and, as we will show later, this is the case at the optimal contract (given we can assume any belief for the out-of-equilibirum message).

- And finally, for each message, the seller's updated belief $p_{i,r}\left(m_r\right)$ has to be consistent with Bayes' rule (BR_r) whenever possible:

$$p_{i,r}\left(m_r\right)\sum_{j \in \Theta} p_{j,r+1} q_j\left(m_r\right) = p_{i,r+1} q_i\left(m_r\right). \tag{2.4}$$

It follows that the seller's problem with imperfect commitment is given by:

$$V_r(p_{r+1}) = \underset{\{q_r, p_r, \Gamma_r\}}{Max} \sum_{i \in \Theta} \sum_{m_r \in M_r} p_{i,r+1} q_i\left(m_t\right)\left(v_r(m_r) + \delta V_{r-1}\left(p_r\left(m_r\right)\right)\right), \tag{2.5}$$

subject to $(2.2) - (2.4)$.

We say that the outcome (q_r, p_r, Γ_r) is *incentive feasible* if it satisfies (2.2)-(2.4) for all $\theta_i \in \Theta$. Additionally, it is *incentive efficient* if it satisfies (2.5), i.e. the seller chooses the best outcome among all of the incentive feasible ones. An *optimal mechanism* is a mechanism Γ_r that belongs to an incentive efficient outcome (q_r, p_r, Γ_r). Finally, (q_r, p_r, Γ_r) and (q_r', p_r', Γ_r') are *payoffs equivalent* if they leave the seller and the buyer (of every possible type) with the same payoffs, i.e.

$$\sum_{i\in\Theta}\sum_{m_r\in M_r} p_{i,r+1}q_i\left(m_r\right)\left(v_r(m_r)+\delta V_{r-1}\left(p_r\left(m_r\right)\right)\right)=$$
$$\sum_{i\in\Theta}\sum_{m'_r\in M'_r} p_{i,r+1}q'_i\left(m'_r\right)\left(v_r(m'_r)+\delta V_{r-1}\left(p'_r\left(m'_r\right)\right)\right),$$
$$\sum_{m_r\in M_r} q_i\left(m_r\right)\left(u_{i,r}(m_r)+\delta U_{i,r-1}\left(p_r\left(m_r\right)\right)\right)=$$
$$\sum_{m'_r\in M'_r} q'_i\left(m'_r\right)\left(u_{i,r}(m'_r)+\delta U_{i,r-1}\left(p'_r\left(m'_r\right)\right)\right),\quad i\in\{L,H\}.$$

2.2.1 Revelation Principle

In this subsection we show that, i) we can restrict to direct mechanisms, ii) that p_r is always determined by Bayes' rule (consequently there are not out-of-equilibrium beliefs) and, iii) that it is enough to consider a subset of all possible q_r.

A direct mechanism is a mechanism in which the message set is the buyer's type set, i.e. $M_r = \Theta$. In this case, the buyer strategy is to report each type with some probability, i.e., $q_i : \Theta \to [0,1]$, with $\sum_{m_r\in\Theta} q_i(m_r) = 1$. Bester and Strausz (2001) provides a revelation principle for environments with imperfect commitment, including the multistage contracting problem studied here. Based on this revelation principle we can seek a solution of (2.5) using direct mechanisms, i.e.:

Lemma 12 *Let's assume a state p_{r+1}. Any solution (q_r, p_r, Γ_r) for (2.5) has payoffs equivalent to a solution $\left(\hat{q}_r, \hat{p}_r, \hat{\Gamma}_r\right)$ where $\hat{\Gamma}_r$ is a direct mechanism and where the buyer reports his type with positive probability, i.e., $\hat{q}_i(i) > 0\ \forall\ i \in \{L, H\}$.*

Proof. This lemma is a direct application of the Proposition 2 and its corollary at Bester and Strausz (2001). ■

Bester and Strausz (2001) shows that is sufficient for the mechanism designer to consider mechanisms in which the set of messages has equal cardinality to the type space. Moreover, they show that we can associate each message with a type that plays the message with positive probability.

A consequence is that the mechanism designer can be restricted to outcomes (q_r, p_r, Γ_r) where the mechanism has Θ as the message set. Then, as every message that belongs to Θ is reported with positive probability, she can always associate a message with the corresponding type. That is, she asks to every type to report the truth with positive probability, i.e., $q_H > 0$ and $1 - q_L > 0$ where we denote by q_H (q_L) the probability that a high-type buyer (low-type buyer) sends a high-type message.

Notice that this revelation principle differs from the standard one (see Myerson 1981) in that there is no guarantee that the buyer reports his true type with certainty. Even so, truthful reporting is always an optimal strategy for the buyer and he still plays it with positive probability.

Given some mechanism Γ_r, (2.2) requires that any message which is played with positive probability must be optimal for the buyer. From the revelation principle either $q_H = 1$ ($q_L = 0$), in which case (2.2) requires that high-type buyer (low-type buyer) prefers to report the truth, or $q_H < 1$ ($q_L > 0$) in which case (2.2) requires indifference between both messages. Hence, (2.2) can be simplified to:

$$IC_{H,r}: \quad u_{H,r}(h) + \delta U_{H,r-1}(p_r(h)) \geq u_{H,r}(l) + \delta U_{H,r-1}(p_r(l)) \text{ with equality if } 1 - q_H > 0,$$
$$IC_{L,r}: \quad u_{L,r}(l) + \delta U_{L,r-1}(p_r(l)) \geq u_{L,r}(h) + \delta U_{L,r-1}(p_r(h)) \text{ with equality if } q_L > 0,$$

where, from now on, we use h and l to indicate high-type and low-type messages, respectively.[5]

As every message is sent with positive probability by at least one type, (2.4) is always satisfied. As a consequence, the posteriors are completely determined by Bayes' rule and p_r is a redundant variable of optimization.

Additionally, we can concentrate our analysis on those incentive feasible outcomes such that $q_H \geq q_L$. For those incentive feasible outcomes such that $q_L > q_H$, we can simply define a new mechanism in which the role of each message is interchanged.

Lemma 13 *All incentive feasible outcomes* (q_r, p_r, Γ_r)*, where* Γ_r *is a direct mechanism, such that* $q_L > q_H$ *is payoff equivalent to an incentive feasible outcome* $(\hat{q}_r, \hat{p}_r, \Gamma_r)$*, with the same direct mechanism* Γ_r*, such that* $\hat{q}_H > \hat{q}_L$*.*

Proof. See the Appendix. ■

Notice that $q_H \geq q_L$ implies $p_{H,r}(h) \geq p_{H,r+1} \geq p_{H,r}(l)$ by Bayes' rule.

2.3 Optimal Selling Mechanism

2.3.1 Road Map

In this section we solve the seller's problem at (2.5), proving that price posting (see Chapter 1 for its definition) is the optimal selling mechanism for every period when $r > 2$.

First, we simplify the problem at (2.5) as in the two-period case (Lemma 3). We show that $IC_{H,r}$ and $IR_{L,r}$ are binding at the optimum, that $IR_{H,r}$ is redundant and that $IC_{L,r}$ can be replaced by a new constraint (SMC_r) which is more useful in the analysis.

Second, we define the continuation values when the discount factor is arbitrarily large. Next, we prove they are well defined (Lemma 4 and Lemma 5) and that they have some particular properties that are going to be useful to solve the seller's problem (from Lemma 6 to Lemma 9).

Finally, we show that the optimal mechanism follows these continuation values and, at the same time, that price posting is the optimal selling mechanism (Theorem 1 and Corollary 1).

2.3.2 Analysis

To solve the seller's problem at (2.5), it is useful to simplify it first. Next lemma establishes the equivalence between (2.5) after simplifications of Section 2.1 and a reduced program.

[5] Notice that when the buyer is indifferent between both messages, he randomizes between them. The seller knows this but she does not observe which probability the buyer chooses for each message. We assume that she can always select the best equilibrium between all the possible ones as is usual in mechanism design.

Lemma 14 *At any period r, the seller's problem at (2.5) is equivalent to*

$$\underset{\{q_r,\Gamma_r\}}{Max} \sum_{i\in\Theta} \sum_{m_r=l,h} p_{i,r+1} q_i(m_r) \left[v_r(m_r) + \delta V_{r-1}(p_r(m_r))\right], \quad \textit{subject to,} \tag{2.6}$$

$$IC^*_{H,r}: \quad u_{H,r}(h) + \delta U_{H,r-1}(p_r(h)) = u_{H,r}(l) + \delta U_{H,r-1}(p_r(l)),$$

$$IR^*_{L,r}: \quad u_{L,r}(l) + \delta U_{L,r-1}(p_r(l)) = 0,$$

$$SMC_r: \quad x_r(h) - x_r(l) \geq \frac{\delta}{\Delta\theta}\left[U_{H,r-1}(p_r(l)) - U_{H,r-1}(p_r(h))\right], \quad \textit{with equality if } q_L > 0,$$

$$BR_r: \quad p_{i,r}(m_r) = \frac{p_{i,r+1} q_i(m_r)}{\sum_{k=L,H} p_{k,r+1} q_k(m_r)}, \; m_r = l, h,$$

$$x_r \in [0,1], \; q_H > 0, \; q_L < 1, \; q_H \geq q_L.$$

Proof. See the Appendix. ■

The interpretation of (2.6) is the same than the one for the reduced program in Chapter 1.

Since we consider the case with only two types, the vector $p_r(m_r)$ is completely determined by $p_{H,r}(m_r)$. From now on, and when it is not explicitly indicated in a different way, we refer as p to the prior of observing a high-type buyer at period r, and $p(m_r)$ to its posterior when a message m_r is sent.

One further simplification is to substitute $w_r(h)$ and $w_r(l)$ into (2.6) using $IR^*_{L,r}$ and $IC^*_{H,r}$. This is, we substitute

$$\begin{aligned} w_r(l) &= x_r(l)\theta_L + \delta U_{L,r-1}(p_r(l)), \\ w_r(h) &= (x_r(h) - x_r(l))\theta_H + x_r(l)\theta_L + \delta U_{L,r-1}(p_r(l)) + \delta U_{H,r-1}(p_r(h)) - \delta U_{H,r-1}(p_r(l)), \end{aligned}$$

into the seller's problem and we get:

$$\begin{aligned} &\underset{\{q_r,x_r\}}{Max} \; W_r(x_r, q_r, p, p(m_r)) \text{ subject to,} \\ &SMC_r, \; BR_r, \\ &x_r \in [0,1], \; q_H > 0, \; q_L < 1, \; q_H \geq q_L. \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} W_r(x_r, q_r, p, p(m_r)) \;=\;& x_r(l)\theta_L + \rho_H(x_r(h) - x_r(l))\theta_H + \delta U_{L,r-1}(p_r(l)) + \\ & \delta\rho_H\left[U_{H,r-1}(p(h)) - U_{H,r-1}(p(l))\right] + \delta\rho_H V_{r-1}(p(h)) + \delta(1-\rho_H)V_{r-1}(p(l)), \end{aligned}$$

and ρ_H is equal to $(pq_H + (1-p)q_L)$.

Continuation Values

We propose some functions for the seller $\tilde{V}_r(p)$ and for the high-type buyer $\tilde{U}_r(p)$, defining them recursively. For low-type buyer, we propose a function which is equal to zero for every p. We show later that they correspond with the equilibrium continuation values.

From Chapter 1 let $\tilde{V}_r(p)$ and $\tilde{U}_r(p)$ equal to $V_r(p)$ and $U_r(p)$ respectively, for periods $r=1$ and $r=2$. Also from Chapter 1, we use notation of τ_1^*, τ_2^* and τ_2.[6] We denote $\tau_1 = \frac{\theta_L}{\theta_H}$ and $\tau_0 = 0$. Then, let:

$$\tilde{V}_r(p) \equiv \begin{cases} p \geq \tau_r & \bar{q}_r(p)p\left(\theta_H + \delta\tilde{V}_{r-1}(1)\right) + \left(1-\bar{q}_r(p)p\right)\delta\tilde{V}_{r-1}(\tau_{r-1}) \\ p \in [\tau_r^*, \tau_r) & q_r^*(p)p\left(\theta_H + \delta\tilde{V}_{r-1}(1)\right) + \left(1-q_r^*(p)p\right)\delta\tilde{V}_{r-1}\left(\tau_{r-1}^*\right) - pq_r^*(p)\delta^{r-1}\Delta\theta \\ p \in [0, \tau_r^*) & \theta_L + \delta\tilde{V}_{r-1}(p) \end{cases},$$

for all $r > 2$;

$$\tilde{U}_r(p) \equiv \begin{cases} p \geq \tau_r & \left(1-\bar{q}_r(p)p\right)\delta\tilde{U}_{r-1}(\tau_{r-1}) \\ p \in [\tau_r^*, \tau_r) & \left(1-q_r^*(p)p\right)\delta\tilde{U}_{r-1}\left(\tau_{r-1}^*\right) + \delta^{r-1}\Delta\theta \\ p \in [0, \tau_r^*) & \theta_L + \delta\tilde{U}_{r-1}(p) \end{cases},$$

for all $r > 2$, where,

- τ_r is the value of $p \in (\tau_{r-1}, 1)$ such that first two lines of $\tilde{V}_r(p)$ coincides and τ_r^* is the value of $p \in \left(\tau_{r-1}^*, 1\right)$ such that last two lines of of $\tilde{V}_r(p)$ coincides.[7]

- $q_r(p, q_L) \equiv \frac{p-\tau_{r-1}}{p(1-\tau_{r-1})} + \frac{(1-p)q_L\tau_{r-1}}{p(1-\tau_{r-1})}$ $\forall p \in (\tau_{r-1}, 1)$, i.e., suppose a low-type buyer is sending a message h with probability q_L, then $q_r(p, q_L)$ is the probability that a high-type buyer sends a message h such that the seller's posterior, when she observes a message l, is equal to τ_{r-1}.

- $\bar{q}_r(p) \equiv \frac{p-\tau_{r-1}}{p(1-\tau_{r-1})}$ $\forall p \in (\tau_{r-1}, 1)$, i.e., the previous probability for the particular case of a low-type buyer sending a message h with zero probability ($q_L = 0$).

- $q_r^*(p) \equiv \frac{p-\tau_{r-1}^*}{p\left(1-\tau_{r-1}^*\right)}$ $\forall p \in \left(\tau_{r-1}^*, 1\right)$, for $r > 2$ and $q_2^*(\tau_2^*) = 1$.

Next figure illustrates seller's value functions, as we shall show later. Intervals $p \in \left[0, \frac{\theta_L}{\theta_H}\right]$, $p \in [\tau_r^*, \tau_r]$ and $p \in [\tau_r, 1]$ are linear in p. The interval $p \in \left[\frac{\theta_L}{\theta_H}, \tau_r^*\right]$ is piecewise linear in p. The figure also illustrates cutoffs τ_r^* and τ_r. These points guarantee that $\tilde{V}_r(p)$ is continuous.

[6] Recall $\tau_1^* = \frac{\theta_L}{\theta_H}$, $\tau_2^* = \frac{\theta_L}{\theta_H}$ and $\tau_2 = \frac{\theta_L[\theta_H+\delta\Delta\theta]}{\theta_H[\theta_L+\delta\Delta\theta]}$.

[7] This is, τ_r is the value of $p \in (\tau_{r-1}, 1)$ such that

$$\bar{q}_r(p)p\left(\theta_H + \delta\tilde{V}_{r-1}(1)\right) + \left(1-\bar{q}_r(p)p\right)\delta\tilde{V}_{r-1}(\tau_{r-1}) = q_r^*(p)p\left(\theta_H + \delta\tilde{V}_{r-1}(1)\right) + \left(1-q_r^*(p)p\right)\delta\tilde{V}_{r-1}(\tau_{r-1}^*) - pq_r^*(p)\delta^{r-1}\Delta\theta,$$

and τ_r^* is the value of $p \in (\tau_{r-1}^*, 1)$ such that

$$q_r^*(p)p\left(\theta_H + \delta\tilde{V}_{r-1}(1)\right) + \left(1-q_r^*(p)p\right)\delta\tilde{V}_{r-1}(\tau_{r-1}^*) - pq_r^*(p)\delta^{r-1}\Delta\theta = \theta_L + \delta\tilde{V}_{r-1}(p).$$

Then, points τ_r and τ_r^* guarantee continuity of $\tilde{V}_r(p)$ on p.

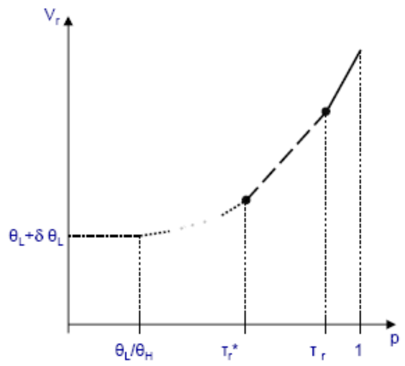

Figure 1: Solid-Line: first line in definition of $\tilde{V}_r(p)$; Dash-Line: second line in definition of $\tilde{V}_r(p)$; Dot-Line: third line in definition of $\tilde{V}_r(p)$.

Previous definition of $\tilde{V}_r(p)$ requires, to be complete, that τ_r^* and τ_r exist and are unique. The following two lemmas prove these properties.

Lemma 15 $\tau_r^* = \frac{\theta_L}{\theta_H}\sum_{i=0}^{r-2}\left(\frac{\Delta\theta}{\theta_H}\right)^i$ *and verifies* $\tau_r^* = \frac{\theta_L}{\theta_H q_r^*(\tau_r^*)}$ $\forall r > 1$.

Proof. See the Appendix. ■

Lemma 16 *Solution* τ_r *exists and it is unique.*

Proof. See the Appendix. ■

Notice that τ_r^* and τ_r are increasing in r.

Next, we propose a functional form for our conjecture of the continuation values. The proof is by induction.

Lemma 17 $\tilde{U}_r(p)$ *and* $\tilde{V}_r(p)$ *verify:*

$$\tilde{U}_r(p) = \Delta\theta\left(\sum_{i\in\Omega_r(p)}\delta^i + \mathbf{I}_{[0,\tau_r)}(p)\delta^{r-1}\right),$$

$$\tilde{V}_r(p) = \theta_L\sum_{i\in\Omega_r(p)}\delta^i + p\theta_H\sum_{i\in\bar{\Omega}_r(p)}\hat{q}_{r-i}(p)\delta^i + \theta_L\mathbf{I}_{[0,\tau_r)}(p)\delta^{r-1} + p\theta_H\mathbf{I}_{[\tau_r,1]}(p)\delta^{r-1},$$

where

$$\hat{q}_{r-i}(p) \equiv \begin{cases} \bar{q}_{r-i}(p) & if \quad p \geq \tau_r \\ q_{r-i}^*(p) & o.w. \end{cases},$$

$$\Omega_r(p) \equiv \left\{i \in \{0,1,...,r-2\} : p \in \left[0, \tau_{r-i}^*\right)\right\},$$

$$\bar{\Omega}_r(p) \equiv \{i \in \{0,1,...,r-2\} \setminus \Omega_r(p)\}.$$

Proof. See the Appendix. ■

As we will show, the set $\Omega_r(p)$ is the set of periods up to $r = 2$ in which the seller sells with probability one no matter the message observed. Its complementary $\bar{\Omega}_r(p)$ is when this does not happen. In particular, $\bar{\Omega}_r(p)$ is the set of periods in which the seller only sells to the high-type buyer with probability $\hat{q}_{r-i}(p)$.

The next lemma ensures that $\tau_r > \tau_r^*$.

Lemma 18 *If δ is sufficiently closed to* 1*, then $\tau_r^* \in (\tau_{r-2}, \tau_{r-1})$ $\forall r > 2$.*

Proof. See the Appendix. ■

Besides,

Lemma 19 *Suppose $U_{r-1}(p) = \tilde{U}_{r-1}(p)$ and $V_{r-1}(p) = \tilde{V}_{r-1}(p)$. If $\delta \in (\delta^*(T), 1)$, then either $\Omega_{r-1}(p(h)) = \Omega_{r-1}(p(l))$ or $\Omega_{r-1}(p(h)) = \Omega_{r-1}(p(l)) \backslash \max\{i \in \Omega_{r-1}(p(l))\}$, where $\delta^*(T)$ is the unique solution in $(0, 1)$ to $\delta^{T-2}(1+\delta) = 1$.*

Proof. See the Appendix. ■

Lemma 8 follows from the facts that τ_r^* is increasing in r and that δ is arbitrarily large. The former implies that $\Omega_r(p)$ is decreasing in p and $\Omega_{r-1}(p(l)) \geq \Omega_{r-1}(p(h))$ since $p(h) \geq p(l)$. The latter implies $\Omega_{r-1}(p(h)) = \Omega_{r-1}(p(l))$ or $\Omega_{r-1}(p(h)) = \Omega_{r-1}(p(l)) \backslash \max\{i \in \Omega_{r-1}(p(l))\}$ in order to verify the SMC.

As in Chapter 1, we say that a mechanism induces learning when the buyer's continuation values are different for each message. Learning becomes relevant when it induces the seller to propose in the future a different mechanism for each message observed in the current period. This implies that buyer's payoffs are different for each message. Notice that this definition is with respect to our conjecture on continuation values.

Definition 20 *A mechanism at period r induces* learning *when $\tilde{U}_{r-1}(p(l)) - \tilde{U}_{r-1}(p(h)) \neq 0$.*

Notice that, since $p(h) \geq p(l)$ and $\tilde{U}_r(p)$ is decreasing in p by definition, *learning* means $\tilde{U}_{r-1}(p(l)) > \tilde{U}_{r-1}(p(h))$. We distinguish the following cases of *learning* and *no-learning* that correspond with Lemma 8.

Lemma 21 Learning *can arise in the following cases:*

- Learning-a*: if $\Omega_{r-1}(p(l)) = \Omega_{r-1}(p(h))$, $p(h) \geq \tau_{r-1}$ and $p(l) \in [\tau_{r-1}^*, \tau_{r-1})$.*
- Learning-b*: if $\Omega_{r-1}(p(h)) = \Omega_{r-1}(p(l)) \backslash \max\{i \in \Omega_{r-1}(p(l))\}$ and $\Omega_{r-1}(p) = \Omega_{r-1}(p(h))$.*
- Learning-c*: if $\Omega_{r-1}(p(h)) = \Omega_{r-1}(p(l)) \backslash \max\{i \in \Omega_{r-1}(p(l))\}$ and $\Omega_{r-1}(p) = \Omega_{r-1}(p(l))$.*

Besides, if there is no-learning*, then $\Omega_{r-1}(p(l)) = \Omega_{r-1}(p(h))$.*

Proof. See the Appendix ■

By application of Lemma 6 and Lemma 9 we have the following remark.

Remark 22 *In* learning-a *$\tilde{U}_{r-1}(p(h)) = 0$ and $\tilde{U}_{r-1}(p(l)) = \delta^{r-2}\Delta\theta$. In* learning-b *and* learning-c*, $\tilde{U}_{r-1}(p(l)) - \tilde{U}_{r-1}(p(h)) = \delta^j \Delta\theta$ where $j = \max\{i \in \Omega_{r-1}(p(l))\}$.*

Optimality

Next, we solve the problem at (2.7) using our conjecture of continuation values and we show that the optimal solution follows that conjecture. At the same time, we prove that the optimal selling mechanism is price posting.

Then, the seller's problem is

$$\begin{aligned}&\underset{\{q_r,x_r\}}{Max}\ \tilde{W}_r\left(x_r,q_r,p,p(m_r)\right)\ \text{subject to,}\\&SMC_r,\ BR_r,\\&x_r\in[0,1]\,,\ q_H>0,\ q_L<1,\ q_H>q_L.\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}\tilde{W}_r\left(x_r,q_r,p,p(m_r)\right) &= x_r(l)\theta_L+\rho_H\left(x_r(h)-x_r(l)\right)\theta_H+\delta\rho_H\left[\tilde{U}_{r-1}\left(p(h)\right)-\tilde{U}_{r-1}\left(p(l)\right)\right]+\\&+\delta\rho_H\tilde{V}_{r-1}(p(h))+\delta(1-\rho_H)\tilde{V}_{r-1}(p(l)),\end{aligned}$$

and ρ_H is equal to $(pq_H+(1-p)q_L)$.

As we did in Chapter 1, we split (2.8) into two subproblems. We consider the two variables maximization problem as a maximization problem in which the seller chooses first q_r and next x_r.[8] This is, fixing q_r, we maximize with respect to x_r. Since seller's payoff are increasing in $x_r(h)$ and the increment of $x_r(h)$ relaxes the SMC_r, then the optimal $x_r(h)$ is 1. On the other hand, the optimal allocation for message l depends on $\rho_H=pq_H+(1-p)q_L$, i.e. $x_r(l)=\hat{x}_r(l,q_r)$ where

$$\hat{x}_r(l,q_r)=\begin{cases}0 & if\quad \rho_H\geq\frac{\theta_L}{\theta_H}\\ \mu & if\quad \rho_H<\frac{\theta_L}{\theta_H}\end{cases},\tag{2.9}$$

with $\mu=\min\left\{1,1-\delta\frac{\tilde{U}_{r-1}(p(l))}{\Delta\theta}+\delta\frac{\tilde{U}_{r-1}(p(h))}{\Delta\theta}\right\}$ when $q_L=0$,[9] and

$$\hat{x}_r(l,q_r)=1-\delta\frac{\tilde{U}_{r-1}\left(p\left(l\right)\right)}{\Delta\theta}+\delta\frac{\tilde{U}_{r-1}\left(p\left(h\right)\right)}{\Delta\theta}\tag{2.10}$$

when $q_L\neq 0$.

---

[8] We are using the general property $\underset{\{x,y\}}{Max}f(x,y)=\underset{\{x\}}{Max}\left(\underset{\{y\}}{Max}f(x,y)\right)$.

[9] The optimal allocation for next period is

$$\hat{x}_{r-1}(l,q_{r-1})=\begin{cases}0 & if & \rho_{H,r-1}>\frac{\theta_L}{\theta_H}\\ \alpha_{r-1} & if & \rho_{H,r-1}=\frac{\theta_L}{\theta_H}\\ \mu & if & \rho_{H,r-1}<\frac{\theta_L}{\theta_H}\end{cases}$$

with $\alpha_{r-1}\in[0,\mu]$. Bester and Strausz specification allows the possibility of giving to the seller the option, at period r, of choosing α_{r-1}. Incluiding this action for the seller complicates the model without upsetting our result. We assume $\alpha_{r-1}=0$. Given this assumption, we can also assume without loss of generality that $\hat{x}_r(l,q_r)=0$ when $\rho_H=\frac{\theta_L}{\theta_H}$ at period r.

Now, we have to solve the seller's maximization problem with respect to q_r, i.e.

$$\begin{aligned}
&\underset{\{q_r\}}{max}\ \tilde{W}_r\left(\hat{x}_r(l,q_r),q_r,p,p(m_r)\right),\ \text{subject to,}\\
&p(h)=\frac{pq_H}{pq_H+(1-p)q_L},\\
&p(l)=\frac{p(1-q_H)}{p(1-q_H)+(1-p)(1-q_L)},\\
&q_H\in(0,1]\,,\ q_L\in[0,1)\,,\ q_H>q_L.
\end{aligned} \tag{2.11}$$

To solve the second subproblem, we differentiate those cases where $x_r(l)=0$ and where $x_r(l)\neq 0$.

Definition 23 *We say that a mechanism has* SMC non-binding *if* $x_r(l)=0$ *and* SMC binding *if* $x_r(l)\neq 0$.

In both cases, it is possible to have *learning* or *no-learning*. Since $x_r(h)=1$ and δ is arbitrarily large, is not possible to have $x_r(l)=0$ at (2.10). It follows that the allocation $x_r(l)=0$ occurs only when $q_L=0$ and $\rho_H\geq\frac{\theta_L}{\theta_H}$ from (2.9). On the other hand, $x_r(l)\neq 0$ occurs either when $q_L=0$ and $\rho_H<\frac{\theta_L}{\theta_H}$, or when $q_L\neq 0$. In both cases, by (2.9) or (2.10), respectively, $x_r(l)=1-\delta^{r-1}$ when there is *learning-a*, $x_r(l)=1-\delta^{j+1}$ when there is *learning-b* or *learning-c*, and $x_r(l)=1$ when there is *no-learning*.

We can use previous terminology to distinguish eight subcases: *SMC non-binding with no-learning* (*SMC+NL*), *SMC non-binding with learning* (*SMC+L*) of cases a, b and c (*SMC+La*, *SMC+Lb* and *SMC+Lc*), *SMC binding with no-learning* (*SMC*+NL*), and *SMC binding with learning* (*SMC*+L*) of cases a, b and c (*SMC*+La*, *SMC*+Lb* and *SMC*+Lc*). Some of them could be empty for some prior. To analyze each subcase we assume that continuation values have the functional form proposed at Lemma 6. Next, we prove that the optimal mechanisms give payoffs that indeed follows our proposal. We also characterized the optimal mechanism for any prior. This is stated in the following theorem.

Theorem 24 *For any* $r>2$ *and for any* $\delta\in(\delta^*(T),1)$, *the continuation payoffs associated to the optimal selling mechanism are such that* $U_r(p)=\tilde{U}_r(p)$ *and* $V_r(p)=\tilde{V}_r(p)$. *The optimal selling mechanism is characterized by:*

- *if* $p\geq\tau_r$, (SMC non-binding with no-learning) *satisfies that* $x_r(h)=1$, $x_r(l)=0$, $w_r(h)=\theta_H$, $w_r(l)=0$, $q_H=\bar{q}_r(p)$, *and* $q_L=0$.

- *if* $p\in[\tau_r^*,\tau_r)$, (SMC non-binding with learning) *satisfies that* $x_r(h)=1$, $x_r(l)=0$, $w_r(h)=\theta_H-\delta^{r-1}\Delta\theta$, $w_r(l)=0$, $q_H=q_r^*(p)$, *and* $q_L=0$.

- *if* $p\in[0,\tau_r^*)$, (SMC binding with no-learning) *satisfies that* $x_r(h)=x_r(l)=1$, $w_r(h)=w_r(l)=\theta_L$, $q_H=q_L\neq 0$.

Proof. We start by assuming that continuation values for period $r-1$ are $\tilde{U}_{r-1}(p)$ and $\tilde{V}_{r-1}(p)$ for high-type buyer and for the seller respectively. We assume zero continuation value for low-type buyer.

We proceed as follow. First, in each of the following claims we get payoffs for the optimal mechanism in each subcase, indicating under which priors the subcase is not empty. These payoffs are either linear or piecewise linear in p. Second, we show that *SMC+NL* and *SMC+L* give the same payoffs at prior $p = \tau_r$ and that the former is steeper than the latter. Third, we show that *SMC*+La*, *SMC+L* and *SMC*+NL* give the same payoffs at $p = \tau_r^*$. By slope comparison, we prove that *SMC*+La* is either dominated by *SMC+L* or by *SMC*+NL*. Finally, *SMC*+Lb* and *SMC*+Lc* are dominated by *SMC*+NL*.

Claim 25 *Optimization of (2.11) subject to the additional constraint* SMC non-binding with no-learning *(*SMC+NL*) verifies that*

$$\begin{aligned} U_r(p) &= 0, \\ V_r(p) &= p\theta_H \sum_{i=0}^{r-2} \delta^i \bar{q}_{r-i}(p) + \delta^{r-1} p\theta_H, \end{aligned} \tag{2.12}$$

with $q_H = \bar{q}_r$ and $q_L = 0$. Moreover, it is defined for $p \geq p^*$ where $p^* = \tau_{r-1} + (1 - \tau_{r-1})\frac{\theta_L}{\theta_H}$.

Proof of Claim 1. In this case, $\tilde{U}_{r-1}(p(l)) - \tilde{U}_{r-1}(p(h)) = 0$ by no-learning and $x_r(l) = 0$ by non-binding. This allocation implies $q_L = 0$ and $\rho_H \geq \frac{\theta_L}{\theta_H}$ from (2.9), requiring $q_H \geq \frac{\theta_L}{\theta_H p}$. On the other hand, $p(h) = 1$ by BR_r and, from the functional form of continuation values at Lemma 6, to have no-learning it must be that $p(l) \geq \tau_{r-1}$, requiring $q_H \leq \bar{q}_r$ by definition of $\bar{q}_r$. Hence $p \geq p^*$ where $p^* = \tau_{r-1} + (1 - \tau_{r-1})\frac{\theta_L}{\theta_H}$. Since $p(l) \geq \tau_{r-1}$ then $p(l) > \tau_{r-1}^*$ and $\Omega_{r-1}(p(h)) = \Omega_{r-1}(p(l)) = \emptyset$. Using previous information, we can get agent's continuation values after substituting it in their functional form at Lemma 6. Plugging them into (2.11) and after some simplifications, the seller maximizes her payoffs with $q_H = \bar{q}_r$ (the maximum q_H such that $p(l) = \tau_{r-1}$), getting (2.12). ■

Claim 26 *Optimization of (2.11) subject to the additional constraint* SMC non-binding with learning *(*SMC+L*) verifies that*

$$\begin{aligned} U_r(p) &= 0, \\ V_r(p) &= p\theta_H \sum_{i=0}^{r-2} \delta^i q^*_{r-i}(p) + \delta^{r-1} \theta_L, \end{aligned} \tag{2.13}$$

with $q_H = q_r^*$ and $q_L = 0$. Moreover, it is defined for $p \geq \tau_r^*$ and only learning-a *is feasible.*

Proof of Claim 2. In this case, $\tilde{U}_{r-1}(p(l)) - \tilde{U}_{r-1}(p(h)) > 0$ by learning and $x_r(l) = 0$ by non-binding. This implies $q_L = 0$ and $\rho_H \geq \frac{\theta_L}{\theta_H}$ from (2.9), requiring $q_H \geq \frac{\theta_L}{\theta_H p}$. By BR_r, $p(h) = 1$ (i.e. $p(h) > \tau_{r-1}$). *Learning-a* is the only learning case which is feasible with $p(h) > \tau_{r-1}$, i.e. $\Omega_{r-1}(p(h)) = \Omega_{r-1}(p(l) = \emptyset$ and $p(l) \in [\tau_{r-1}^*, \tau_{r-1})$. Then, $q_H \leq q_r^*$ by definition of q_r^*, and jointly with $q_H \geq \frac{\theta_L}{\theta_H p}$, implies that p must be larger or equal to $\tau_{r-1}^* + (1 - \tau_{r-1}^*)\frac{\theta_L}{\theta_H}$ which it turns to be equal to τ_r^* by Lemma 4. After substituting previous conditions in the functional form of continuation values at Lemma 6, plugging them into (2.11) and after some simplifications, the seller maximizes her payoffs with $q_H = q_r^*$ (the maximum q_H such that $p(l) = \tau_{r-1}^*$), getting (2.13). ■

Claim 27 *Optimization of (2.11) subject to the additional constraint* SMC binding with no-learning *(*SMC*+NL*) verifies that*

$$\begin{aligned} U_r(p) &= \Delta\theta + \delta\tilde{U}_{r-1}(p), \\ V_r(p) &= \theta_L + \delta\tilde{V}_{r-1}(p), \end{aligned} \tag{2.14}$$

with $q_H = q_L \neq 0$.

Proof of Claim 3. In this case, $\tilde{U}_{r-1}(p(l)) - \tilde{U}_{r-1}(p(h)) = 0$ and $\Omega_{r-1}(p(l)) = \Omega_{r-1}(p(h))$ by *no-learning*. Also, *binding* with *no-learning* means $x_r(l) = 1$ either by (2.9) when $q_L = 0$ and $\rho_H < \frac{\theta_L}{\theta_H}$ or by (2.10) when $q_L \neq 0$.
Since $\Omega_{r-1}(p(l)) = \Omega_{r-1}(p(h))$, operating with the definition of seller's continuation values, we get that $\rho_H\tilde{V}_{r-1}(p(h)) + (1-\rho_H)\tilde{V}_{r-1}(p(l))$ is equal to $\tilde{V}_{r-1}(p)$.[10] Hence, substituting previous conditions into (2.11) and after some simplification, we get that payoffs are equal to (2.14). The seller can choose any q_H and q_L subject to *SMC*+NL*. In particular, let $q_H = q_L \neq 0$ which give $\tilde{V}_{r-1}(p(h)) = \tilde{V}_{r-1}(p(l)) = \tilde{V}_{r-1}(p)$. ■

Claim 28 *Optimization of (2.11) subject to the additional constraint* SMC binding with learning-a *(*SMC*+La*) verifies that the seller's expected payoffs are bounded above by*

$$\theta_L + \sum_{i\in\Omega_{r-1}(p)} \delta^{i+1}\theta_L + pq_H\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p)} \delta^{i+1}q^*_{r-1-i}(p(h)) + p(1-q_H)\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p)} \delta^{i+1}q^*_{r-1-i}(p(l)) + \delta^{r-1}pq_H\theta_H,$$

and bounded below by

$$\theta_L + \sum_{i\in\Omega_{r-1}(p)} \delta^{i+1}\theta_L + pq_H\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p)} \delta^{i+1}\bar{q}_{r-1-i}(p(h)) + p(1-q_H)\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p)} \delta^{i+1}\bar{q}_{r-i-1}(p(l)) + \delta^{r-1}pq_H\theta_H.$$

for the optimal q_H *such that* $p(h) \geq \tau_{r-1}$ *and* $p(l) \in [\tau^*_{r-1}, \tau_{r-1})$. *This mechanism is defined for* $p \geq \tau^*_{r-1}$. *Moreover, when* $p = \tau^*_r$ *seller's expected payoffs are equal to (2.14) with* $q_H = q^*_r(\tau^*_r)$ *and* $q_L = 0$.

Proof of Claim 4. Now $\tilde{U}_{r-1}(p(l)) - \tilde{U}_{r-1}(p(h)) \neq 0$ by *learning* and $x_r(l) \neq 0$ by *binding*. As consequence $\hat{x}_r(l, q_r) < 1$ from (2.9) when $q_L = 0$ and $\rho_H < \frac{\theta_L}{\theta_H}$ or, from (2.10) when $q_L \neq 0$. Since we are considering *learning-a*, $\Omega_{r-1}(p(l)) = \Omega_{r-1}(p(h))$, $p(h) \geq \tau_{r-1}$ and $p(l) \in [\tau^*_{r-1}, \tau_{r-1})$, giving $\tilde{U}_{r-1}(p(h)) = 0$ and $\tilde{U}_{r-1}(p(l)) = \delta^{r-2}\Delta\theta$, i.e. $x_r(l) = 1 - \delta^{r-1}$.
By definition, $p(h) \geq \tau_{r-1}$ and $p(l) \in [\tau^*_{r-1}, \tau_{r-1})$ implies that $\hat{q}_{r-1-i}(p(h))$ is equal to $\bar{q}_{r-1-i}(p(h))$

[10] *No-learning* implies that $\hat{q}_{r-1-i}(p(h))$ and $\hat{q}_{r-1-i}(p(l))$ are either equal to $\bar{q}_{r-1-i}(p(h))$ and to $\bar{q}_{r-1-i}(p(l))$ respectively, or equal to $q^*_{r-1-i}(p(h))$ and to $q^*_{r-1-i}(p(l))$. Then,

$$\rho_H p(h)\hat{q}_{r-1-i}(p(h)) + (1-\rho_H)p(l)\hat{q}_{r-1-i}(p(l)) = p\hat{q}_{r-1-i}(p).$$

and $\hat{q}_{r-1-i}(p(l))$ to $q^*_{r-1-i}(p(l))$. From $\Omega_{r-1}(p(l)) = \Omega_{r-1}(p(h))$ it follows $\bar{\Omega}_{r-1}(p(l)) = \bar{\Omega}_{r-1}(p(h))$ and $\Omega_{r-1}(p) = \Omega_{r-1}(p(h))$. We can get continuation values form Lemma 6, and after substituting them at (2.11) and some simplifications, the seller has to choose (q_H, q_L) to maximize,

$$\begin{aligned}\theta_L + \sum_{i\in\Omega_{r-1}(p)} \delta^{i+1}\theta_L + pq_H\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p)} \delta^{i+1}\bar{q}_{r-1-i}(p(h)) \\ + p(1-q_H)\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p)} \delta^{i+1}q^*_{r-1-i}(p(l)) + \delta^{r-1}pq_H\theta_H,\end{aligned} \tag{2.15}$$

subject to $p(h) \geq \tau_{r-1}$, $p(l) \in \left[\tau^*_{r-1}, \tau_{r-1}\right)$.

Notice that, since $p(h) \geq p \geq p(l)$, this mechanism can only be defined for $p \geq \tau^*_{r-1}$, which implies $\Omega_{r-1}(p) = \emptyset$.
Since $\bar{q}_r(\cdot) < q^*_r(\cdot)$ by definition, (2.15) is bounded above when replacing $\bar{q}_{r-1-i}(p(h))$ by $q^*_{r-1-i}(p(h))$.[11] On the other hand, (2.15) is bounded below when replacing $q^*_{r-i-1}(p(l))$ with $\bar{q}_{r-i-1}(p(l))$.

Moreover, when $p = \tau^*_r$, the seller maximizes (2.15) choosing $q_H = q^*_r(\tau^*_r)$ (in order to $p(l) = \tau^*_{r-1}$), and $q_L = 0$ (to get $p(h) = 1$ while $\rho_H < \frac{\theta_L}{\theta_H}$) making

$$\theta_L + \theta_L \sum_{i\in\bar{\Omega}_{r-1}(\tau^*_r)} \delta^{i+1} + \tau^*_r(1 - q^*_r(\tau^*_r))\theta_H \sum_{i\in\bar{\Omega}_{r-1}(\tau^*_r)} \delta^{i+1}\frac{\tau^*_{r-1} - \tau^*_{r-2-i}}{\tau^*_{r-1}\left(1 - \tau^*_{r-2-i}\right)} + \delta^{r-1}\theta_L.$$

Using the relation of τ^*_r with τ^*_{r-1} implicit in Lemma 4 we get that

$$\theta_L + \tau^*_r(1 - q^*_r(\tau^*_r))\theta_H\frac{\tau^*_{r-1} - \tau^*_{r-2-i}}{\tau^*_{r-1}\left(1 - \tau^*_{r-2-i}\right)} = \tau^*_r\theta_H q^*_{r-1-i}(\tau^*_r).$$

Then, seller's maximum payoffs can be written as

$$\theta_L + \tau^*_r\theta_H \sum_{i\in\bar{\Omega}_{r-1}(\tau^*_r)} \delta^{i+1}q^*_{r-1-i}(\tau^*_r) + \delta^{r-1}\theta_L.$$

This last expression is equivalent to seller's payoff at (2.14) when we replace in it the functional form of $\tilde{V}_{r-1}(\tau^*_r)$ from Lemma 6. By the definition of $\tilde{V}_r(p)$, it is also equal to (2.13) for $p = \tau^*_r$. ■

Claim 29 *Optimization of (2.11) subject to the additional constraint* SMC *binding with learning-b (*SMC*+Lb) verifies that the seller expected payoffs are equal to (2.14) with (q_H, q_L) such that q_H is equal to $\frac{p-\tau^*_{r-j-2}}{p(1-\tau^*_{r-j-2})} + \frac{(1-p)q_L\tau^*_{r-j-2}}{p(1-\tau^*_{r-j-2})}$, where $j = \max\{i \in \Omega_{r-1}(p(l))\}$. This mechanism is defined for $p < \tau_{r-1}$.*

Proof of Claim 5. $\tilde{U}_{r-1}(p(l)) - \tilde{U}_{r-1}(p(h)) \neq 0$ by *learning* and $x_r(l) \neq 0$ by *binding*. As

[11] To simplify (2.15) we use

$$\begin{aligned}\rho_{H,}p(h)\hat{q}_{r-1-i}(p(h)) + (1-\rho_H)p(l)\hat{q}_{r-1-i}(p(l)) &= \\ &= pq_H\hat{q}_{r-1-i}(p(h)) + p(1-q_H)\hat{q}_{r-1-i}(p(l)) \\ &= p\hat{q}_{r-1-i}(p),\end{aligned}$$

consequence $\hat{x}_r(l,q_r) < 1$ from (2.9) when $q_L = 0$ and $\rho_H < \frac{\theta_L}{\theta_H}$ or, from (2.10) when $q_L \neq 0$. Since we are considering *learning-b*, $|\Omega_{r-1}(p\,(l))| - |\Omega_{r-1}(p\,(h))| = 1$ and $\Omega_{r-1}(p) = \Omega_{r-1}(p\,(h))$, with $p\,(h) < \tau_{r-1}$ and $p\,(l) < \tau_{r-1}$, giving $\tilde{U}_{r-1}\,(p(l)) - \tilde{U}_{r-1}\,(p(h)) = \delta^j \Delta\theta$ where $j = \max\{i \in \Omega_{r-1}(p\,(l))\}$, i.e. $x_r(l) = 1 - \delta^{j+1}$.

Since $\tau_{r-1} > p\,(h) \geq p\,(l)$ and $p\,(h) \geq p \geq p\,(l)$, this mechanism is defined for $p < \tau_{r-1}$. Additionally, $\tau_{r-1} > p\,(h) \geq p\,(l)$ implies $\hat{q}_{r-1-i}(p(\cdot)) = q^*_{r-1-i}\,(p\,(\cdot))$ by definition. Let j to be the larger $i \in \Omega_{r-1}\,(p\,(l))\,$, i.e.$\Omega_{r-1}\,(p\,(l)) = \{0,1,...,j\}$ and $\bar{\Omega}_{r-1}\,(p\,(l)) = \{j+1,...,r-3\}$. By definition of $\Omega_{r-1}\,(p\,(l))$, it must be that $p\,(l) \in \left[\tau^*_{r-2-j}, \tau^*_{r-1-j}\right)$, and since $|\Omega_{r-1}\,(p\,(l))| = |\Omega_{r-1}\,(p\,(h))| + 1$, $p\,(h) \in \left[\tau^*_{r-1-j}, \tau^*_{r-j}\right)$ with $\Omega_{r-1}\,(p\,(h)) = \{0,1,...,j-1\}$ and $\bar{\Omega}_{r-1}\,(p\,(h)) = \{j,j+1,...,r-3\}$. Continuation values for $r-1$ are given by Lemma 6. Substituting continuation values and allocations at (2.11) and after some simplifications,[12] the seller chooses (q_H, q_L) to maximize

$$\theta_L + \sum_{i\in\Omega_{r-1}(p(h))} \theta_L \delta^{i+1} + \delta^{j+1}\left(pq_H - \frac{(1-p)q_L\tau^*_{r-j-2}}{1-\tau^*_{r-j-2}}\right)\theta_H + $$
$$+ p\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p(h))\setminus\{j\}} q^*_{r-1-i}\,(p)\,\delta^{i+1} + \theta_L\delta^{r-1}$$
$$\text{subject to } p\,(h) \in \left[\tau^*_{r-1-j}, \tau^*_{r-j}\right),\ p\,(l) \in \left[\tau^*_{r-2-j}, \tau^*_{r-1-j}\right).$$

These payoffs are maximized with q_H equal to $\frac{p-\tau^*_{r-j-2}}{p(1-\tau^*_{r-j-2})} + \frac{(1-p)q_L\tau^*_{r-j-2}}{p(1-\tau^*_{r-j-2})}$ (which is the maximum q_H such that $p\,(l) \in \left[\tau^*_{r-2-j}, \tau^*_{r-1-j}\right)$, i.e. $p\,(l) = \tau^*_{r-2-j}$), making

$$\theta_L + \theta_L \sum_{i\in\Omega_{r-1}(p)} \delta^{i+1} + p\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p)} q^*_{r-1-i}\,(p)\,\delta^{i+1} + \theta_L\delta^{r-1}.$$

These payoffs are equal to the expression at (2.14) when we replace $\tilde{V}_{r-1}(p)$ by its functional form defined for $p < \tau_r$ at Lemma 6. ■

Claim 30 *Optimization of (2.11) subject to the additional constraint* SMC binding with learning-c *(*SMC*+Lb*) verifies that the seller expected payoffs are equal to*

$$\theta_L + \theta_L \sum_{i\in\Omega_{r-1}(p)\setminus\{j\}} \delta^{i+1} + p\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p)\cup\{j\}} q^*_{r-1-i}\,(p)\,\delta^{i+1} + \theta_L\delta^{r-1}, \tag{2.16}$$

with (q_H, q_L) such that q_H is equal to $\frac{p-\tau^*_{r-j-2}}{p(1-\tau^*_{r-j-2})} + \frac{(1-p)q_L\tau^*_{r-j-2}}{p(1-\tau^*_{r-j-2})}$, where $j = \{\max i \in \Omega_{r-1}(p\,(l))\}$. This mechanism is defined for $p < \tau_{r-1}$.

Proof of Claim 6. $\tilde{U}_{r-1}\,(p(l)) - \tilde{U}_{r-1}\,(p(h)) \neq 0$ by *learning* and $x_r(l) \neq 0$ by *binding*. As consequence $\hat{x}_r(l,q_r) < 1$ from (2.9) when $q_L = 0$ and $\rho_H < \frac{\theta_L}{\theta_H}$ or, from (2.10) when $q_L \neq 0$. Since we

[12] Notice that

$$\rho_{H,}p\,(h)\,q^*_{r-j-1}\,(p\,(h)) = pq_H - \frac{(1-p)q_L\tau^*_{r-j-2}}{1-\tau^*_{r-j-2}}.$$

are considering *learning-c*, $|\Omega_{r-1}(p\,(l))| - |\Omega_{r-1}(p\,(h))| = 1$ and $\Omega_{r-1}(p) = \Omega_{r-1}(p\,(l))$, with $p\,(h) < \tau_{r-1}$ and $p\,(l) < \tau_{r-1}$, giving $\tilde{U}_{r-1}\,(p(l)) - \tilde{U}_{r-1}\,(p(h)) = \delta^j \Delta\theta$ where $j = \max\{i \in \Omega_{r-1}(p\,(l))\}$, i.e. $x_r(l) = 1 - \delta^{j+1}$.
Since $\tau_{r-1} > p\,(h) \geq p\,(l)$ and $p\,(h) \geq p \geq p\,(l)$, this mechanism is defined for $p < \tau_{r-1}$.
Let j to be the larger $i \in \Omega_{r-1}\,(p\,(l))$. By definition of $\Omega_r(p)$, $p\,(l) \in \left[\tau^*_{r-2-j}, \tau^*_{r-1-j}\right)$ and, since $\Omega_{r-1}(p) = \Omega_{r-1}\,(p\,(l))$, also $p \in \left[\tau^*_{r-2-j}, \tau^*_{r-1-j}\right)$. Following the same procedure than in previous point, seller's maximum payoffs are equal to (2.16).
■

We have the optimal mechanisms for each subcase. We proceed now to compare them. Notice that (2.12) and (2.13) are linear on p and, that (2.14), payoffs at Claim 4 and (2.16) are piecewise linear in p with slopes increasing in p.[13]

Notice that (2.12) is the functional form at Lemma 6 defined for $p \geq \tau_r$ and (2.13) the one for $p < \tau_r$, i.e. $V_r(p) = \tilde{V}_r(p)$ and $U_r(p) = \tilde{U}_r(p)$. Then, they follow our definition of $\tilde{V}_r(p)$ and $\tilde{U}_r(p)$ for $p \geq \tau_r$ and $p \in [\tau^*_r, \tau_r)$ respectively. By this definition, they are equal at $p = \tau_r$. Finally, (2.12) is steeper than (2.13) due to $\frac{1}{1-\tau_{r-1-i}} \geq \frac{1}{1-\tau^*_{r-1-i}}$. Then, (2.12) dominates (2.13) when $p \geq \tau_r$ and the opposite when $p < \tau_r$.

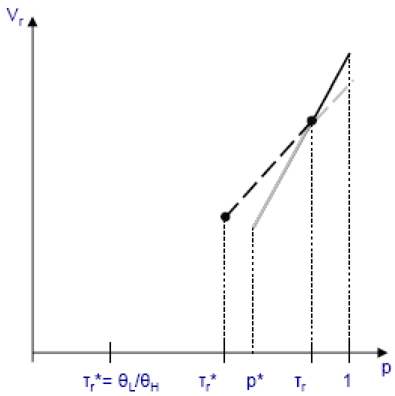

Figure 2: Maximum Seller's payoffs. Dash-Line: SMC non-binding with learning; Solid-Line: SMC non-binding with no-learning.

Payoffs at (2.14) have the functional form at Lemma 6 defined for $p < \tau^*_r$, i.e. $V_r(p) = \tilde{V}_r(p)$ and $U_r(p) = \tilde{U}_r(p)$. Then by the definition of $\tilde{V}_r(p)$ and $\tilde{U}_r(p)$, (2.14) and (2.13) are equal at $p = \tau^*_r$. From Claim 4, (2.14) and (2.13) are also equal to seller's payoffs under *SMC binding with learning-a* at $p = \tau^*_r$.

When $p \in \left(\tau^*_{r-1}, \tau_r\right)$, the slope of (2.14) is bounded above by $\sum_{i=1}^{r-2} \delta^i \theta_H \frac{1}{\left(1-\tau^*_{r-1-i}\right)}$ which is lower than the one of (2.13), equal to $\theta_H \frac{1}{\left(1-\tau^*_{r-1}\right)} + \sum_{i=1}^{r-2} \delta^i \theta_H \frac{1}{\left(1-\tau^*_{r-1-i}\right)}$. When $p < \tau^*_{r-i}$ for $i \in \{1, ..., r-2\}$ the slope of (2.14) is decreasing in i.[14] Then, the current mechanism dominates the one

[13] When $p \in \left[\tau^*_{r-2-j}, \tau^*_{r-1-j}\right)$, $\bar{\Omega}_{r-1}(p) = \{j+1, ..., r-3\}$ by definition. Applying functional form for continuation values at Lemma6, $\tilde{V}_{r-1}(p)$ at (2.14) has slope $\sum_{i=j+1}^{r-3} \delta^i \theta_H \frac{1}{1-\tau^*_{r-1-i}}$. On the other hand, when $p \in \left[\tau^*_{r-1-j}, \tau^*_{r-j}\right)$, now $\bar{\Omega}_{r-1}(p) = \{j, ..., r-3\}$, and $\tilde{V}_{r-1}(p)$ has a larger slope equal to $\delta^j \theta_H \frac{1}{1-\tau^*_{r-1-j}} + \sum_{i=j+1}^{r-3} \delta^i \theta_H \frac{1}{1-\tau^*_{r-1-i}}$. The same argument can be used to check that slopes are increasing in p in payoffs at Claim 4 and at (2.16).

[14] When $p < \tau^*_2$, the slope of (2.14) is zero.

under *SMC binding with learning* when $p < \tau_r^*$ and the opposite when $p \in [\tau_r^*, \tau_r]$. When $p > \tau_r$, the slope of (2.14) is now equal to $\theta_H \sum_{i=1}^{r-2} \delta^i \frac{1}{(1-\tau_{r-1-i})} + \delta^{r-1}\theta_H$, which is lower than the one of (2.12).

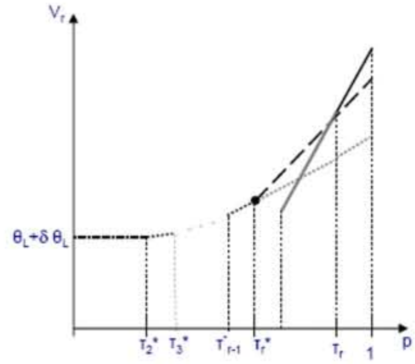

Figure 3: Maximum Seller's payoffs. Dot-Line: SMC binding with no-learning; Dash-Line: SMC non-binding with learning.

From Claim 4, the upper bound of seller's payoffs has a maximum slope equal to

$$\theta_H \sum_{i=0}^{r-3} \delta^{i+1} \frac{1}{\left(1-\tau^*_{r-2-i}\right)} + \delta^{r-1}\theta_H,$$

when assuming that, in the maximization of (2.11), the seller could choose $q_H = 1$ such that $p(h) \geq \tau_{r-1}$, $p(l) \in \left[\tau^*_{r-1}, \tau_{r-1}\right)$. This slope is lower to the one at (2.13) which is $\theta_H \sum_{i=0}^{r-2} \delta^i \frac{1}{\left(1-\tau^*_{r-1-i}\right)}$.[15] On the other hand, the lower bound of seller's payoffs has a minimum slope equal to

$$\theta_H \sum_{i=0}^{r-3} \delta^{i+1} \frac{1}{\left(1-\tau_{r-2-i}\right)} + \delta^{r-1}\theta_H,$$

when assuming that, in the maximization of (2.11), the seller could choose $q_H = 1$ such that $p(h) \geq \tau_{r-1}$, $p(l) \in \left[\tau^*_{r-1}, \tau_{r-1}\right)$. This slope is larger than the one of (2.14) (bounded above by $\theta_H \sum_{i=1}^{r-2} \delta^i \frac{1}{(1-\tau_{r-1-i})}$) when $p < \tau_r^*$. Then, when $p > \tau_r^*$, a mechanism *SMC binding with learning-a* is dominated by a mechanism *SMC non-binding with learning* and, when $p < \tau_r^*$, it is dominated by *SMC binding with no-learning*.

[15] $\frac{1}{\left(1-\tau^*_{r-1}\right)} + \frac{\delta}{\left(1-\tau^*_{r-2}\right)} + ... + \frac{\delta^{r-2}}{\left(1-\tau^*_1\right)} > \frac{\delta}{\left(1-\tau^*_{r-2}\right)} + ... + \delta^{r-1}$ due to $\frac{1}{\left(1-\tau^*_{r-1}\right)} > 1$.

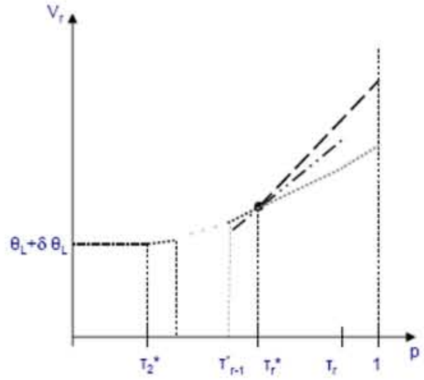

Figure 4: Maximum Seller's payoffs. SMC binding with learning-a (Dash-Double Dot-Line) dominated by SMC binding with no-learning (Dot-Line) and SMC non-binding with learning (Dash-Line)

From Claim 5 *SMC binding with learning-b* gives the same payoffs than *SMC binding with no-learning*. From Claim 6, and *SMC binding with learning-c* is weakly dominated by *SMC binding with no-learning* since payoffs at (2.16) are lower than payoffs at (2.14) due to $pq^*_{r-1-j}(p) < \frac{\theta_L}{\theta_H}$ when $p < \tau^*_{r-1-j}$ by Lemma 4.

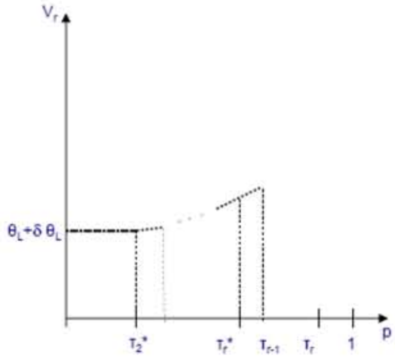

Figure 5: SMC binding with no-learning coincides with SMC binding with learning-b for $p < \tau_{r-1}$.

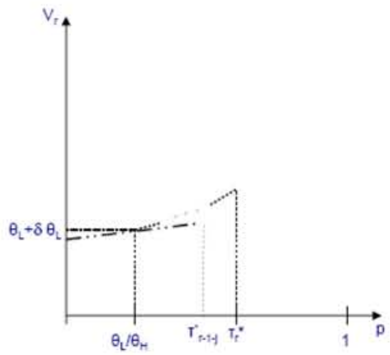

Figure 6: Dot-Line: SMC binding with no-learning; Dash-Line: SMC binding with learning-c.

Concluding, the optimal mechanism is a *SMC non-binding with no-learning* when $p \geq \tau_r$, a *SMC non-binding with learning* when $p \in [\tau^*_r, \tau_r)$ and a *SMC binding with no-learning* when $p \in [0, \tau^*_r)$.

Optimal allocations are

$$x_r(h) = 1, \;\; x_r(l) = \begin{cases} 0 & \text{if } p \geq \tau_r^* \\ 1 & \text{if } p < \tau_r^* \end{cases} .$$

Optimal payments are obtained by replacing, for each case, allocations and continuation values at $IR_{L,r}^*$ and $IC_{H,r}^*$ and solving for $w_r(l)$ and $w_r(h)$,

$$w_r(h) = \begin{cases} \theta_H & \text{if } p \geq \tau_r \\ \theta_H - \delta^{r-1}\Delta\theta & \text{if } p \in [\tau_r^*, \tau_r) \\ \theta_L & \text{if } p < \tau_r^* \end{cases} , \quad w_r(l) = \begin{cases} 0 & \text{if } p \geq \tau_r^* \\ \theta_L & \text{if } p < \tau_r^* \end{cases} .$$

Notice that low type payoffs are zero with previous $w_r(l)$ given that we assumed zero continuation values for him. ■

The argument of the proof relies on the following: the optimal payoffs for each subcase are either linear or piecewise linear functions of p. The upper envelope of these functions only contains *SMC non-binding with no-learning* (when $p \geq \tau_r$), *SMC non-binding with learning-a* ($p \in [\tau_r^*, \tau_r)$) and *SMC binding with no-learning* ($p \in [0, \tau_r^*)$). Then, this upper envelope characterizes the optimal mechanism for every prior and it is equal to the definition of $\tilde{V}_r(p)$. It is summarized in Figure 1.

Optimal mechanisms in Theorem 1 are direct mechanisms with allocation $x_r(l) \in \{0, 1\}$. We state in the following corollary that the optimal direct mechanism can be implemented by a price posting, which is an indirect mechanism. To do that, we propose an alternative outcome $(\hat{q}_r, \hat{p}_r, \hat{\Gamma}_r)$ where $\hat{\Gamma}_r$ is a price posting mechanism and we check that this outcome is payoff equivalent to the incentive efficient outcome (q_r, p_r, Γ_r) that solves (2.6) and contains the optimal selling mechanism characterized in the theorem. Since the proof is mechanic, we relegate it to the Appendix.

Corollary 31 *When $r > 2$, the optimal selling mechanism can be implemented by a price posting equal to*

i) θ_H when $p \geq \tau_r$, the high-type buyer randomizes and the low-type buyer never buys;

ii) $\theta_H - \delta^{r-1}\Delta\theta$ when $p \in [\tau_r^*, \tau_r)$, The high-type buyer always buys and the low-type buyer never buys and;

iii) θ_L when $p < \tau_r^*$, both types always buy.

Proof. See the Appendix. ■

When the seller is optimistic ($p \geq \tau_r^*$), she offers a price posting that separates types. This is, only the high-type buyer buys with positive probability. In case of being extremely optimistic ($p \geq \tau_r$), the seller offers a price posting equal to θ_H. The high-type buyer randomizes and, in case of not buying, the seller will ask for a price equal to θ_H in the following period again. Then, she exploits the buyer extracting all his surplus in every period. This exploiting case corresponds with *SMC non-binding with no-learning*. In case of being moderately optimistic ($p \in [\tau_r^*, \tau_r)$), the seller offers a price posting equal to $\theta_H - \delta^{r-1}\Delta\theta$. Now, the seller is bribing the high-type buyer to induce him to reveal his type. This bribe is equal to his future discounted losses by being discriminated in the current period. This bribing case corresponds with *SMC non-binding with learning*. Finally, when the seller is pessimistic ($p < \tau_r^*$), she offers a price equal to θ_L. This is the pooling case, when both buyer types always buy, which corresponds with *SMC binding with no-learning*.

Belief's Dynamic

Figure 7 indicates how beliefs evolve. Starting at an optimistic prior (i.e. $p \geq \tau_r^*$), the seller's beliefs are updated gradually as information is revealed when the buyer does not buy. On the other hand, when the buyer buys, she quickly learns that she is facing a high-type consumer with certainty. Starting at $p \in \left[\frac{\theta_L}{\theta_H}, \tau_r^*\right)$, seller's beliefs are not updated up to some period $r-i$ where $p \geq \tau_{r-i}^*$. When $p < \frac{\theta_L}{\theta_H}$, seller's beliefs are never updated.

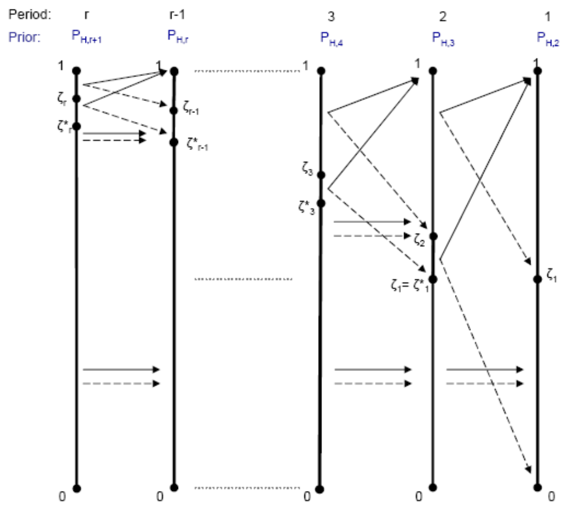

Figure 7: Belief dynamic under different priors for T>2 periods.. A full line shows how beliefs evolve when the buyer buys the good. The dash line is when he does not buy.

2.4 Concluding Remarks

This paper generalizes the model of Chapter 1 for many periods when both players have the same discount factor. It proves that within this framework the optimal selling procedure is to post a price in every period. The paper also gives a complete characterization of equilibrium payoffs.

A natural extension is to study which is the optimal mechanism when discount factors are different but close to one.

Chapter 3

Labor Mobility and Technology Choice

(joint work with Daniel García-González)

3.1 Introduction

High labor turnover constitutes a common feature in many industries. Workers often climb the job ladder by moving to new firms who bid up their wages. Indeed, the availability of workers performing similar tasks in other firms is one of the classical reasons for industry agglomeration. As Marshall (1890) pointed out "a localized industry gains a great advantage from the fact that it offers a constant market for skill". This is a very important feature of high-skilled sectors like high-tech or consulting. For instance, software developers in Silicon Valley have very high mobility rates (see Mukherjee (2008)).

In this "market for experience", poaching firms have an important informational disadvantage with respect to the initial employer. They make wage offers to prospective workers based on partial (if any) information regarding their past performance. The current employer will often be more informed about the worker's ability, and may, therefore, profit from such informational advantage. In particular, if all firms had the same underlying productivity, an extreme form of adverse selection obtains and all workers would remain with their initial employers.

If, on the other hand, there is heterogeneity among firms, more productive firms may prefer to hire experienced workers from rival firms.[1] Less productive firms may then try to retain their best workers by concealing information about their skill. For instance by using different technologies they may be able to adjust the amount of public information about workers skill. The goal of this paper is to understand this interaction by embedding a career-concern model into a very simple labor market framework. In particular, we assume that past performance is perfectly observable but is only an imperfect signal of workers' ability. The signal-to-noise ratio depends on the characteristics of the tasks, which are chosen by the current employer. We show that some firms decide to design inefficiently the characteristics of the task in order to conceal information from the market, and decrease the likelihood of losing skilled workers.

[1] In our model, we assume that the maximum expected productivity is obtained when the more productive worker is assigned to the more capitalized firm.

More precisely, we consider an industry populated by overlapping generations of two-period lived workers who vary in their skill level. Firms are infinitely lived and require one worker per period to produce output. Firms are also heterogenous, differing in their capital stock (i.e., their marginal labor productivity) and their task configuration, which jointly determine the expected productivity of a given worker in the firm.[2] Idle firms post vacancies and decide whether to search young workers from the unemployment pool or workers currently employed in a rival firm. In the latest case, they are randomly matched with a young and successful worker and make her an offer. Incumbent firms have then the right to match the offer or let the worker leave for the new employer.

We characterize the steady state equilibrium of the industry. In equilibrium, there is a unique capital stock such that more capitalized firms decide to poach workers from fellow employers and firms with lower capital stocks will hire unemployed workers at their reservation wage. The poaching game has a unique equilibrium in un-dominated strategies where the poaching firm only makes an offer if it will not be matched by the current employer independently of the skill of the worker.

We also show that in equilibrium neither the most nor the least capitalized firms will distort their task allocation. The first group are never poached by a rival firm while the second group cannot avoid it. On the other hand, those firms in the middle of the distribution may distort their task allocation to deter poaching. By committing themselves to inefficient technologies, firms retain their successful workers with higher probability by reducing the probability that idle firms assign him to be a high-skill worker. Thus, firms increase the ex-ante expected output of an old worker. Due to linearity, the former effect is independent of the capital stock as long as the firm is able to deter some poaching firms. On the other hand, the latter effect is increasing in the capital stock of the firm, because more capitalized firms are able to retain more workers in the second period. Thus, the use of inefficient technologies is more prominent in bigger firms within this range of the distribution. Eventually, however, as firms become more productive they are able to retain their best workers without distorting their technology and so they choose the efficient task combination.

3.1.1 Related literature

Most previous literature analyze the effect of the disclosure mechanism in the equilibrium of the market. For example, in Wolitzky (2012) prospect employers lack information about previous output realization but the current employer may submit performance reports. He studies the effect of allowing for secret contracts between the worker and the current employer.

Koch (2009) studies a model under which the employer proposes different contracts to her workers. The initial employer has two employees, one talented and one ordinary. After production takes place, workers go to the labor market and other firms form beliefs about their abilities observing their earnings. In this framework, the optimal contract never reveals perfectly workers' skills.

The closest paper to us is Mukherjee (2008), who presents a career-concerns model with mobility across firms and incumbent advantage. In his model the incumbent and "raider" firms are determined ex-ante and both raiders are homogenous and compete for the worker in the second period. The incumbent firm can commit whether to disclose her private information (at no cost). Most of the analysis is similar to that of the present paper, but our industry-equilibrium framework offers new insights on the endogenous productivity distribution of firms and the equilibrium assignment of

[2] We assume that this capital stock is set once and for all periods and is observable to rival firms.

workers to firms. In particular, we show that the inefficiencies associated with adverse selection and inefficient matching are exacerbated due to the endogenous selection of firms into incumbents and raiders.

3.2 Model

Environment:

There is an infinite number of periods. At every period, the economy is populated by a unit measure of firms and an incoming cohort of workers, with measure $N > 1$. Firms live infinitely. They are identified by their capital stock $k \sim U[\underline{k}, \overline{k}]$. Each firm (she) has a production technology and has to hire one worker to produce with this technology (explained below). Workers (he) live for two periods. At the first period of their lives workers are young and, in the second period they become old. They differ in their skills $\theta \in \{\theta_L, \theta_H\}$, with $\theta_L = 0$ and $\theta_H = 1$. At any moment in time, the proportion of high-skill workers in the incoming cohort of workers is equal to $\alpha \in (0, 1)$. Firms and workers discount the future with the same discount factor δ. In each period, a firm's payoff is given by the difference between her production and her worker's salary while the worker's payoff is given by his salary. All players maximize their expected payoff.

Firm's Technology:

Each firm chooses her own technology. Parameters p and q characterize this technology in the following way. If a firm with capital stock k hires a high-skill worker, she produces an output equal to k with probability p when the worker is young. When this worker becomes old, the firm produces an output equal to k with probability 1. On the other hand, if this firm hires a low-skill worker, output k is realized with probability q when he is young and with probability 0 when he is old.[3] The set of feasible technologies verifies that $p + \quad q = \bar{p}$, with $p \in [\frac{\bar{p}}{1+\quad}, \bar{p}]$ and $q \in [0, \frac{\bar{p}}{1+\quad}]$. We assume that $\quad > \frac{1-\alpha}{\alpha}$, so the firm chooses $(\bar{p}, 0)$ in a one-period game.[4] This technology is meant to capture different ways in which firms may organize production, giving salience to better workers. For instance, a firm may use very standardized procedures (which all workers are able to learn) or very innovative ones (which offer a high skill premium).[5] To describe the technology choosing, we use the function $p : [\underline{k}, \bar{k}] \to [\frac{\bar{p}}{1+\quad}, \bar{p}]$ (and $q(k) = \frac{\bar{p}-p(k)}{\quad}$).

Timing:

At time $t = 0$ all firms choose (publicly) their technologies. This technology remains constant during all the game.

In any other period $t \geq 1$ firms are either idle (without a worker) or active (with a worker hired in previous period). Active firms observe the skill of her own worker after production takes place while idle firms do not observe worker's skill at other firms. However, idle firms do observe if workers at active firms have been successful in production or not.

Next, an idle firm decides whether to make an offer to a successful worker or hiring an unemployed

[3] Since $\theta_L = 0$ and $\theta_H = 1$, technology in second period can also be interpreted that the firm produces an output equal to $k\theta_L$ and $k\theta_H$ respectively.

[4] Notice that is the slope of the technology frontier and $\frac{1-\alpha}{\alpha}$ the slope of the isoquants of the firms. Therefore, when $\quad > \frac{1-\alpha}{\alpha}$, the optimal technology in a one-period game is at the corner where $p \geq q$

[5] One could interpret p and q as measuring the elasticity of substitution across high and low skilled labor in different technologies

worker.[6] In the former case, the idle firm, which was randomly match with an active firm with a successful worker, makes a take-it-or-leave-it offer to the worker. In case the offer is rejected, remain idle for the rest of the period. The active firm has the right to match the offer and keep the successful worker. We assume throughout that firing costs are prohibitive. Then, unsuccessful workers will remain at the active firm.[7]

On the other hand, all active firms that lose their worker and all idle firms that do not look for successful workers at rival firms are randomly match with a young worker at the unemployment pool. Since employers are able to distinguish young and old workers and there is excess supply of young workers ($N > 1$), old workers in the pool will not be hired and thus we assume that they leave it to move to another industry.

Old workers that leave the industry and those who die after production in an active firm, are replaced by young workers with the same skill level. These new young workers join the unemployment pool.

Strategies

At the beginning of the game (at $t = 0$), the strategy of each firm is to choose $p(k)$.

At every period $t \geq 1$, a strategy (s_t, w_t) of an idle firm is a function $s_t : [\underline{k}, \bar{k}] \to \{U, P\}$ which specifies the action of hiring from the pool U or poaching from a rival firm P and a payment for poaching $w_t : [\underline{k}, \bar{k}]^2 \to \Re$. A strategy $(\hat{s}_t, \hat{w}_t)$ of an active firm is a function $\hat{s}_t : [\underline{k}, \bar{k}] \to \{C, NC\}$ which specifies the action of making a counter offer C or not NC and a a payment $\hat{w}_t$ for the counter offer $\hat{w}_t : [\underline{k}, \bar{k}]^2 \times \{\theta_L, \theta_H\} \to \Re$. Workers decide whether to accept or not an offer. Since we assume that workers are subject to a limited liability constraint and that outside options for workers give them zero payoffs, when they are young they accept any non-negative salary and when they are old they accept the best offer.

3.3 Analysis

In this section we study properties of a Steady State Equilibrium (defined below).[8] In particular we prove that it is in cutoff strategies and that some firms which go the the unemployment pool for workers will distort their technology to deter poaching while others will not.

We proceed as follow. First, we assume that firms have chosen their optimal technologies at $t = 0$. We also assume that a Steady State Equilibrium exists. Before studying its properties, in next subsection we solve for wages when an idle firm decides to poach from an active firm. Next, we defined our Steady State Equilibrium and we prove that it is in cutoff strategies. Finally, we solve for $p(k)$ at $t = 0$.

[6] Since $p(k) \geq q(k)$, no output is always bad news about the worker is high-skill. Hence, firms that decided to poach will target their search only to successful workers.

[7] Endogenous firing of unsuccessful workers would complicate the analysis without adding new insights.

[8] To find equilibriums different to a steady state equilibrium (e.g. at $t = 1$ or $t = 2$), it would be necessary to track the distributions of poaching firms and the distributions of firms that go to the pool for each previous period. This task seems not very tractable.

3.3.1 Poaching: Solving for Wages

We now simplify the game by eliminating weakly un-dominated wages in the poaching game. This is, in this subsection, we want to know which are the offers that the idle firm has to make in order to be successful in poaching.

We assume that firm's continuation values (when going to the pool or when poaching) are bounded above, increasing in θ and increasing in k.

First, notice that, since young workers accept any salary, when a firm is matched with a worker from the unemployment pool she will offer a salary equal zero to him.

Now, we consider when an idle firm with capital k is matched with a worker at an active firm with a capital stock k^I. The idle firm makes an offer to the worker. The active firm decides whether to match or not. If the offer is not matched, the worker leaves for the new firm. Otherwise the worker stays. In the former case, the active firm has to go to the pool for a new worker. Let's denote $V^U(k^I)$ to her continuation values.

Since the idle firm can direct its matching to successful young workers, then the active firm production was k^I when she had chosen (p^I, q^I). The expected output of this worker in the idle firm is $k\alpha_{k^I,1}$, where $\alpha_{k^I,1}$ is the probability that the worker is high-skill conditional on successful in production when he was at firm k^I, and on k^I firm failing to match the offer. On the other hand, the expected output of the worker in the active firm is k^I if he is a high-skill worker or 0 if he is a low-skill worker, where this output is conditional on the succeed of the active firm in matching the offer.

Thus, any offer $\omega > 0$ will not be matched by the active firm if her worker is low-skill because the active firm prefers going to the unemployment pool. In case that the active firm has a high-skill worker, a wage equal to $k^I - (1-\delta)V^U(k^I)$ makes the active firm indifferent between making the counteroffer equal to this amount or going to the unemployment pool for a new worker. So, if an idle firm makes an offer lower to that indifference wage, the active firm prefers to match the offer, keep her worker and going to the pool tomorrow than going to the pool today. Then, any offer $\omega < k^I - (1-\delta)V^U(k^I)$ is dominated by offering $\omega = 0$ with which the idle firm succeed in poaching a low-skill worker only. However, unemployed workers in the pool may be high skilled with positive probability and accept any non-negative wage. Hence, poaching low-skill workers is dominated by hiring unemployed workers from the pool.

On the other hand, offering $k^I - (1-\delta)V^U(k^I)$ will not be matched in any case. This offer guarantees to the idle firm to get a high-skill worker but also a low-skill one. Thus, the unique equilibrium outcome in un-dominated strategies is to offer k^I if and only if $k\alpha_{k^I,1} \geq k^I$.[9]

Lemma 32 *If $\delta \to 1$, in equilibrium, an idle firm with capital stock k that chooses to poach and it is matched with an active firm with capital stock k^I makes an offer if and only if $k\alpha_{k^I,1} \geq k^I - (1-\delta)V^U(k^I)$, and the offer it makes is $\omega(k) = k^I - (1-\delta)V^U(k^I)$.*

Proof. Direct from previous explanation. ■

Notice then, that in equilibrium, the active firm never makes a counteroffer.

[9] This equilibrium can be shown to be unique by adding an epsilon cost of making an offer

3.3.2 Steady State Equilibrium

In this Section we introduce the problem and define the Steady State Equilibrium we use to solve it. Since the equilibrium is stationary, a firm that decides to poach (hiring from the pool) is going to do it in every period.

Suppose a firm with capital stock k. In steady state, her expected total payoffs when going to the unemployment pool is

$$\begin{aligned} V^U(k) =& k\Psi(p(k)) + \delta\left(1-\Psi(p(k))\right)\left[k\alpha_{k,0} + \delta V^U(k)\right] \\ &+ \delta\Psi(p(k))\left[k\alpha_{k,1}\phi(k) + \delta V^U(k)\phi(k) + V^U(k)(1-\phi(k))\right], \end{aligned} \tag{3.1}$$

where

$$\Psi(p(k)) = p(k)\alpha + q(k)(1-\alpha), \qquad \alpha_{k,0} = \frac{(1-p(k))\alpha}{1-\Psi(p(k))}, \qquad \alpha_{k,1} = \frac{p(k)\alpha}{\Psi(p(k))},$$

are the probability that the worker is high-skill conditioned on a failure on production in previous period and conditioned on a succeed on production in previous period, respectively, and $(1-\phi(k))$ is the probability of being poached where $\phi(k) : [\underline{k}, \bar{k}] \to [0,1]$.[10]

On the other hand, in steady state, the expected total payoffs of a firm k when decides to poach from a randomly selected active firm that has a successful young worker, is

$$V^P(k) = \int \left(k\alpha_{k^I,1} - k^I\right)^+ \left(\frac{\Psi(p(k^I))Prob(k^I \in B)(\frac{2-\phi(k^I)}{2})}{\int \Psi(p(x))Prob(x \in B)(\frac{2-\phi(x)}{2}))dF(x)}\right) dF(k^I) + \delta V^P(k), \tag{3.2}$$

where

- $\alpha_{k^I,1} = \frac{p(k^I)\alpha}{\Psi(p(k^I))}$ is the probability that the worker is high-skill conditioned on a succeed on production when he was at firm k^I,
- $F(x)$ is the c.d.f. of $x \sim U[\underline{k}, \overline{k}]$,
- $B \equiv \{k : s(k) = U\}$ is the set of firms that hire workers from the unemployment pool,
- $\phi(k^I)$ is the probability that the firm with capital stock k^I is not poached, [11]
- $\frac{2-\phi(k^I)}{2}$, assuming the equilibrium is stationary, this ratio is the proportion of periods with young workers.[12]

Since workers will always choose to work for the highest paying firm and firing costs are prohibitive, these equations completely define the problems faced by each firm, given their initial capital

[10] Since poaching firms can direct their poaching to successful young worker, this probability is positive only if the firm k goes to the pool for a worker and he is young and successful at the moment of being poached.

[11] We do not need an explicit expression for $\phi(k)$ for our analysis. We assume $\phi'(k) > 0$ and $\phi''(k) \leq 0$.

[12] Each firm k^I that goes today to the unemployment pool for a young worker, lose tomorrow its worker with probability $(1-\phi(k))$ and goes again to the pool for another young worker.

stock and their technology choice.[13] Since firms are infinitely lived and population is stationary, we define a Steady State Equilibrium of the Industry as follows.

Definition 33 *Given an optimal choice of $(p(k), q(k))$ by every firm k, a Steady State Equilibrium is a $\{V(k), s(k), \phi(k), \Phi(k)\}$ such that, equilibrium payments are as described above, and*

$$\begin{aligned} &V(k) = \max\left\{V^U(k), V^P(k)\right\}, \\ &s(k) = U, \text{ if } V^U(k) > V^P(k), \\ &s(k) = P, \text{ otherwise}, \\ &\int_{s(k)=U} (1-\phi(k))dF(k) = \int_{s(k)=P} \Phi(k)dF(k), \end{aligned}$$

with $\Phi(k)$ as the probability that the firm with capital stock k poaches from a rival.

The first two equations just says that firms choose whether to search in the pool or poach from another firm optimally, given that the environment is stationary. The last equation is a feasibility constraint that requires that the "supply" and "demand" of poached workers are in equilibrium.[14]

3.3.3 Equilibrium Characterization

In what follows we shall assume that the discount factor approaches 1. We will now show that, if there exists a Steady State equilibrium, it is in cutoff strategies such that $s(k) = U$ for all $k \leq k^*$ and $s(k) = P$ otherwise.

Proposition 34 *Suppose $\delta \to 1$ and $\bar{k} > \frac{\underline{k}}{(1-2\alpha)}$. Then, there exists $k^* \in (\underline{k}, \bar{k})$ such that $s(k) = U$ if and only if $k < k^*$. Otherwise $s(k) = U$ for all k.*

Proof. See the Appendix. ■

This Proposition states that if there is enough heterogeneity across firms, there will be sorting of firms into both markets, with bigger firms choosing to poach workers.

This allows us to rewrite the Equilibrium Condition as

$$\int_{\underline{k}}^{k^*} (1-\phi(k))dF(k) = \int_{k^*}^{\bar{k}} \Phi(k)dF(k)$$

Since all workers who are hired remain in the industry until they die, we have the following equalities. Let S be the measure of workers hired when young, then

$$\begin{aligned} S &= \int_{\underline{k}}^{k^*} \frac{2-\phi(k)}{2} dF(k) \\ &= \int_{k^*}^{\bar{k}} \Phi(k)dF(k) + \int_{\underline{k}}^{k^*} \frac{\phi(k)}{2} dF(k). \end{aligned}$$

[13] Recall that a firm poaches only workers who have been successful in previous period. Then those firms with capital stock k^I belong to the subset of firms that went to the unemployment pool for workers, they have a young worker and have been successful in production.

[14] As we said for function $\phi(k)$, we neither need an explicit expression for $\Phi(k)$. We assume $\Phi'(k) > 0$ and $\Phi''(k) \geq 0$.

The first line of the equation says that the total number of workers hired when young in the industry must equate the total number of firms who has recruited young workers, since each firm who chooses to search in the pool spends $\frac{2-\phi(k)}{2}$ proportion of periods with young workers. The second line says that the total number of old workers equals the total number of firms recruiting from the pool who did not lose their workers plus those firms who poach workers from another firm successfully.

3.4 Technology Choice

The equilibrium structure depends on the technology chosen by all firms at the beginning of the game. Notice that increasing q has three effects. First, q reduces the expected output of a young worker since $> \frac{1-\alpha}{\alpha}$, independently of k. Second, higher q decreases the expected skill of successful workers, thereby deterring poaching. Finally, it increases the expected skill of unsuccessful workers and, therefore, has an ambiguous effect on the output of workers who do not change firms. These last two effects are only relevant for firms who may lose their workers to others in equilibrium. Thus, firms which are not subject to losing their workers to bigger firms have no incentive to distort their productivity.

Analogously, firms in the lower tail of the distribution of capital stocks are unable to deter other firms to poach their successful workers. Nonetheless, they have a relatively unskilled labor force and so may profit from increasing q in order to increase output. This is so if and only if $1-\bar{p} \geq \frac{\alpha\gamma}{1-\alpha} - \frac{2}{\alpha}$.

Finally, for firms in the middle of the distribution all three effects are relevant. The direct effect on young workers and the effect on deterrence are independent of the capital level (within this range). The effect on the expected output of a retained old worker may be non-monotonic in the level of capital because bigger firms have higher probability of retention but also a better average worker. In any case, as firms get bigger, this effect starts to diminish and eventually becomes negative. Thus, big enough firms prefer not to distort their technology.

Proposition 35 *Let* $k_1 \in (\alpha k^*, min\left\{\bar{k}\alpha_{\bar{k},1}, k^*\right\}]$. *When* $\delta \to 1$, *the technology choice as a function of* k *is as follows:*

1. *If* $1-\bar{p} < \frac{\alpha\gamma}{1-\alpha} - \frac{2}{\alpha}$, *then:*

   (a) $p(k) = \bar{p}$ *when* $k \geq k_1$,

   (b) $p(k) = \bar{p}$ *when* $k \leq \alpha k$,

   (c) $p(k) \leq \bar{p}$ *otherwise;*

2. *if* $1-\bar{p} \geq \frac{\alpha\gamma}{1-\alpha} - \frac{2}{\alpha}$, *then:*

   (a) $p(k) = \bar{p}$ *when* $k \geq k_1$,

   (b) $p(k) \leq \bar{p}$ *when* $k < k_1$.

Proof. See the Appendix. ■

3.5 Extensions

3.5.1 Welfare

It may be interesting to study the Social Planner Problem associated with this Industry, in the sense of maximizing total surplus. Only efficient technologies will be chosen in equilibrium. The First-Best is thus defined as an allocation of workers to firms. Some firms are devoted to hire successful workers while the remaining firms hire young workers and retain those that are unsuccessful.

Efficient allocation, which requires all firms to use the efficient technology $\{\bar{p}, 0\}$, is defined by a cutoff k^{**} such that all firms with lower productivity than k^{**} are assigned to recruiting new workers and maintain unsuccessful ones, while the remaining firms recruit successful workers.

The equilibrium cutoff level k^{*} may be too high or too low as compared with k^{**}. If $k^{*} < k^{**}$ the probability of a successful worker moving to a better firm is now higher than the optimal one. On the other hand, if $k^{*} > k^{**}$ there is too little mobility.

3.5.2 Transfer Fees

Adverse-selection leads incumbent employers to distort their technology to obtain rents. A standard way to eliminate the inefficiencies due to the adverse-selection issue is to eliminate the limited liability constraint. Indeed, workers would then be willing to accept negative wages when young in order to buy the opportunity to get promoted. This negative wage is a rent for the employer and it may aligned her incentives with the social optimum. Limited liability and minimum wages are, however, relevant in most interesting applications. In particular, in the markets we described in the Introduction, successful workers may earn orders of magnitude more than unsuccessful ones (or drop-outs). Thus, most workers are unable to finance their bid to show their ability.[15]

A potential solution to this problem is the introduction of transfer fees. If the fee (a transfer from the worker to his current employer) equals the expected output (conditional on success), incumbent firms have no longer interest in acquiring inefficient technologies. This reduces the impact of adverse selection and improves the allocation of workers to firms. Transfer fees are very prominent in some specific environments like soccer, where worker turnover is very high (often from bad teams into better ones, thus fostering assortative matching) and rents for workers may be extremely high. Interestingly, most teams play their best prospects even if this ensures that they will lose them to better teams.

3.6 Conclusions

In this paper we propose a model to explain why some firms could choose technologies that harm their payoffs in the short-term. This is explained by the fact that they compete for high-skill workers with rivals firms. This finding is consistent with anecdotal evidence. We have also propose some interesting extensions.

[15] Interestingly, this is not so in all markets. F1 pilots often pay to drive in smaller teams

Appendix A

Appendix to Chapter 1

A.1 Seller's Sequential Problem

A general model of the seller's sequential problem has the following components.

The initial probability of facing a high-type buyer is denoted by $p_{H,3}$, and for a low-type buyer by $p_{L,3} = 1 - p_{H,3}$.

At every point in time r, we denote as $\bar{y}_{r+1} \equiv (\Gamma_T, m_T, ..., \Gamma_{r+1}, m_{r+1})$ to the history of past actions up to r.

The probability $p_{i,r}$ that the seller assigns to type i -her beliefs- given that she observes history $\bar{y}_r$ is given by $p_{i,r} : \bar{y}_r \to [0, 1]$, and we denote by $p_r(\bar{y}_r)$ to the vector $(p_{L,r}(\bar{y}_r), p_{H,r}(\bar{y}_r))$.

Then at every period, the seller's strategy $\ _r$ is to choose a mechanism Γ_r given the history $\bar{y}_{r+1}$, i.e. $\ _r : \bar{y}_{r+1} \to \Upsilon$, where Υ is the space of mechanisms.

Next, the buyer observes his types i, the history and the mechanism proposed by the seller. His strategy is to send a message $m_r \in M_r$ with probability $q_{i,r}(\cdot)$, $q_{i,r} : M_r \times \Gamma_r \times \bar{y}_{r+1} \to [0, 1]$, for $i \in \{L, H\}$ and that verifies $\sum_{m_r \in M_r} q_i(m_r; \Gamma_r, \bar{y}_{r+1}) = 1$. At next period,the seller updates her beliefs and propose a new mechanism and so on.

Suppose we are in a two period game. Then, at $r = 2$ the seller wants to maximize her expected payoff,

$$\begin{aligned} \underset{\{q_s, p_s, \Gamma_s\}_{s=2}^{1}}{Max} \quad & \sum_{i \in \Theta} p_{i,3} \sum_{m_2 \in M_2} q_{i,2}(m_2, \Gamma_2, \bar{y}_3) [w_2(m_2) + \\ & \delta \sum_{i \in \Theta} p_{i,2}(\bar{y}_2) \sum_{m_1 \in M_1} q_{i,1}(m_1, \Gamma_1, \bar{y}_2) [w_1(m_1)]] \end{aligned} \tag{A.1}$$

subject to $\{q_s, p_s, \Gamma_s\}_{s=2}^{1}$ being PBE implementable.

In order to have a PBE, the buyer's and seller's strategy must be best responses in every period. Given Γ_s, the buyer chooses his reporting strategy anticipating the future seller's beliefs (he maximizes his expected payoff (IC_2)). As response to the buyers strategy, the seller specifies an optimal outcome for next period (SRC_1). Additionally, we have the buyer's participation constraint (IR_2) and beliefs have to be consistent with Baye's Rule (BR_2). Then, the seller's problem at (A.1) is constrained to the following conditions:

$$
\begin{aligned}
IC_{i,2} \quad : \quad & \{q_{i,s}(m_s,\Gamma_s,\bar{y}_{s+1})\}_{s=2}^{1} \in \\
& \underset{\{\tilde{q}_{i,s}(m_s,\Gamma_s,\bar{y}_{s+1})\}_{s=2}^{1}}{\operatorname{argmax}} \sum_{m_2\in M_2} \tilde{q}_{i,2}(m_2,\Gamma_2,\bar{y}_3)\left[u_{i,2}(m_2)+\right. \\
& \qquad \left.\delta \sum_{m_1\in M_1} \tilde{q}_{i,1}(m_1,\Gamma_1,\bar{y}_2)\left[u_{i,1}(m_1)\right]\right], \quad \forall i\in\Theta, \\
IR_{i,2} \quad : \quad & \sum_{m_2\in M_2} q_{i,2}(m_2,\Gamma_2,\bar{y}_3)\left[u_{i,2}(m_2)+\right. \\
& \qquad \left.\delta \sum_{m_1\in M_1} q_{i,1}(m_1,\Gamma_1,\bar{y}_2)\left[u_{i,1}(m_1)\right]\right] \geq 0, \quad \forall i\in\Theta \text{ with } p_{i,3}>0, \\
BR_2 \quad : \quad & p_{i,2}(m_2,\Gamma_2,\bar{y}_3)\sum_{j\in\Theta} p_{j,3}q_{j,2}(m_2,\Gamma_2,\bar{y}_3) = p_{i,3}q_{i,2}(m_2,\Gamma_2,\bar{y}_2), \\
SRC_1 \quad : \quad & \underset{\{q_1,p_1,\Gamma_1\}}{Max} \sum_{i\in\Theta} p_{i,2} \sum_{m_1\in M_1} q_{i,1}(m_1,\Gamma_1,\bar{y}_2)\left[w_1(m_1)\right], \\
& \qquad s.t.: IC_{i,1}\ ,\ IR_{i,1}\ ,\ BR_1.
\end{aligned}
$$

assuming that the reservation utility for every type is equal zero.

Then, the seller chooses the best $\{q_s,p_s,\Gamma_s\}_{s=2}^{1}$ between all of them that are PBE implementable.

In our particular specification, at any period r the prior has all the information that the seller needs to take a decision. Then, we can write the previous problem as a recursive one where p_{r+1} is the state variable at the beginning of each period r.

Suppose we are in the last period $r=1$. Then, the seller solves,

$$
V_1(p_2) = \underset{\{q_1,\Gamma_1\}}{Max} \sum_{i\in\Theta} p_{i,2} \sum_{m_1\in M_1} q_{i,1}(m_1)\, w_1(m_1),
$$

subject to

$$
\begin{aligned}
IC_{i,1} \quad : \quad & q_{i,1}(m_1) \in \underset{\{\tilde{q}_{i,1}(m_1)\}}{\operatorname{argmax}} \sum_{m_1\in M_1} \tilde{q}_{i,1}(m_1)\, u_{i,1}(m_1) \quad \forall i\in\Theta, \\
IR_{i,1} \quad : \quad & \sum_{m_1\in M_1} q_{i,1}(m_1)\, u_{i,1}(m_1) \geq 0 \quad \forall i\in\Theta \text{ with } p_{i,2}>0.
\end{aligned}
$$

In $r=2$,

$$
\begin{aligned}
V_2(p_3) = \underset{\{q_t,p_t,\Gamma_t\}_{t=2}^{1}}{Max} \quad & \sum_{i\in\Theta} p_{i,3} \sum_{m_2\in M_2} q_{i,2}(m_2)\left[w_2(m_2)+\right. \\
& \left.\delta \sum_{i\in\Theta} p_{i,2}(m_2) \sum_{m_1\in M_1} q_{i,1}(m_1)\, w_1(m_1)\right],
\end{aligned}
$$

subject to $IC_{i,2}$, $IR_{i,2}$, BR_2 and SRC_1. Following the Principle of Optimality, previous problem can be written as

$$\begin{aligned}
V_2(p_3) &= \underset{\{q_2,p_2,\Gamma_2\}}{Max} \sum_{i\in\Theta} p_{i,3} \sum_{m_2\in M_2} q_{i,2}(m_2)\left[w_2(m_2)+\delta V_1(p_2)\right], \\
s.t. &: \\
IC_{i,2} &: q_{i,2}(m_2) \in \arg \max_{\{\tilde{q}_{i,2}(m_2)\}} \sum_{m_2\in M_2} \tilde{q}_{i,2}(m_2)\left[u_{i,2}(m_2)+\delta U_{i,1}(m_2)\right] \;\; \forall i\in\Theta, \\
IR_{i,2} &: \sum_{m_2\in M_2} q_{i,2}(m_2)\left[u_{i,2}(m_2)+\delta U_{i,1}(m_2)\right] \geq 0 \;\; \forall i \in \Theta \;\text{ for } i \text{ such that } p_{i,2}>0,
\end{aligned}$$

and BR_2, where $U_{i,1}(m_2) = \sum_{m_1\in M_2} q_{i,1}(m_1)\, u_{i,1}(m_1)$, i.e. the buyer's payoffs given by $IC_{i,1}$.

A.2 Proof of Remark 1

Proof. Consider a message set M_1 with two possible messages $\{"take-it", "leave-it"\}$, a mechanism with an allocation given by

$$x_1(m_1) = \begin{cases} 1 & if \quad m_1 = take-it, \\ 0 & if \quad m_1 = leave-it, \end{cases}, \; m_1 \in M_1,$$

probabilities of observing each message defined by

$$\begin{aligned}
&\hat{q}_i(take-it) \equiv q_i x_1(h) + (1-q_i)x_1(l), \\
&\hat{q}_i(leave-it) \equiv 1 - \hat{q}_i(take-it),
\end{aligned}$$

By Revelation Principle $q_H = 1$ and $q_L = 0$ then $\hat{q}_H(take-it) = 1$, $\hat{q}_L(take-it) = x_1(l)$.

When $p_{H,2} < \frac{\theta_L}{\theta_H}$ the optimal direct selling mechanism has allocations $x_1(l) = 1$, then $\hat{q}_L(take-it) = 1$. Using a price $\hat{w}_1(take-it) = \theta_L$, instant payoffs under both mechanisms are equal for every player.

When $p_{H,3} \geq \frac{\theta_L}{\theta_H}$, $x_1(l) = 0$ and $\hat{q}_L(take-it) = 0$. The optimal direct selling mechanism has payments $w_1(h) = \theta_H$ and $w_1(l) = 0$. Using $\hat{w}_1(take-it) = w_1(h)$, instant payoffs under both mechanisms are equal for every player.

Then, both mechanisms are payoff equivalent for every prior. ■

A.3 Proof of Lemma 2

Proof. Suppose an incentive feasible outcome (q_r, p_r, Γ_r), where Γ_r is a direct mechanism, and with $q_L > q_H$. By the revelation principle, $q_H > 0$ and $q_L < 1$. Since $q_L > q_H$, all IC constraints hold with equality, i.e.

$$\begin{aligned}
IC_{H,r}: \quad & u_{H,r}(h) + \delta U_{H,r-1}(p_r(h)) = u_{H,r}(l) + \delta U_{H,r-1}(p_r(l)), \\
IC_{L,r}: \quad & u_{L,r}(l) + \delta U_{L,r-1}(p_r(l)) = u_{L,r}(h) + \delta U_{L,r-1}(p_r(h)).
\end{aligned}$$

To be incentive feasible also IR and BR have to be satisfied,

$$\begin{aligned}
&IR_{H,r}: \quad u_{H,r}(h)+\delta U_{H,r-1}(p_r(h)) \geq 0, \\
&IR_{L,r}: \quad u_{L,r}(l)+\delta U_{L,r-1}(p_r(l)) \geq 0, \\
&BR_r: \quad p_{H,r}(m_r)\sum_{j\in\Theta} p_{j,r+1}q_j(m_r) = p_{H,r+1}q_H(m_r) \quad \text{with } m_r = l,h.
\end{aligned}$$

The new outcome $(\hat{q}_r, \hat{p}_r, \Gamma_r)$ is created by renaming types such that now, $\hat{q}_H = q_L$ and $\hat{q}_L = q_H$. Then $\hat{q}_H > \hat{q}_L$ as is required. The new constraints are all satisfied, with $IC_{H,r} = \hat{IC}_{L,r}$, $IC_{L,r} = \hat{IC}_{H,r}$, $IR_{H,r} = \hat{IR}_{L,r}$, $IR_{L,r} = \hat{IR}_{H,r}$ and $p_{H,r}(m_r) = \hat{p}_{L,r}(m_r)$.

Also, the seller remains indifferent, i.e.

$$\sum_{i\in\Theta}\sum_{m_r\in\{l,h\}} p_{i,r+1}q_i(m_r)\left[v_r(m_r)+\delta V_{r-1}(p_r(m_r))\right] = \sum_{i\in\Theta}\sum_{m_r\in\{l,h\}} \hat{p}_{i,r+1}\hat{q}_i(m_r)\left[v_r(m_r)+\delta V_{r-1}(\hat{p}_r(m_r))\right].$$

■

A.4 Proof of Lemma 3

Proof. In order to prove this simplification for two periods we follow a similar procedure than in the static case. So, this prove has four steps:

Step 1: $IC_{H,2} + IR_{L,2} \Longrightarrow IR_{H,2}$,

First, notice that $u_{H,2}(m_2) + \delta U_{H,1}(p_2(m_2)) \geq u_{L,2}(m_2) + \delta U_{L,1}(p_2(m_2))$ $\forall m_2$ by $x_2(m_2)\theta_H - w_2(m_2) \geq x_2(m_2)\theta_L - w_2(m_2)$ and by $U_{L,1}(p_2) = 0$ $\forall p_2$ and $U_{H,1}(p_2) \geq 0$ $\forall p_2$, from solution at period $r = 1$.

From previous result, $u_{H,2}(l) + \delta U_{H,1}(p_2(l)) \geq u_{L,2}(l) + \delta U_{L,1}(p_2(l))$. By $IR_{L,2}$, $u_{L,2}(l) + \delta U_{L,1}(p_2(l)) \geq 0$. It follows that $u_{H,2}(h) + \delta U_{H,1}(p_2(h)) \geq 0$ by $IC_{H,2}$, i.e. $IR_{H,2}$ holds.

Step 2: Optimality $\Longrightarrow IR^*_{L,2} + IC^*_{H,2}$,

By optimality we mean that the seller proposes an outcome (q_2, p_2, Γ_2) that maximize her profits.

From step 1, $u_{H,2}(h) + \delta U_{H,1}(p_2(h)) \geq u_{L,2}(l) + \delta U_{L,1}(p_2(l))$.

We now assume that both types start with the same payment $\hat{w}_2$, then

$$x_2(h)\theta_H - \hat{w}_2 + \delta U_{H,1}(p_2(h)) \geq x_2(l)\theta_L - \hat{w}_2 + \delta U_{L,1}(p_2(l)) > 0.$$

In order to improve her payoffs, the seller can increase the payment $\hat{w}_2$ asked to both types by some amount. She continues doing that up to Δw that makes $\hat{x}_2\theta_L - \hat{w}_2 + \delta U_{L,1}(p_2(l)) - \Delta w = 0$. Fixing $w_2(l) = \hat{w}_2 + \Delta w$ we get $IR_{L,2}$ is binding. Note that $IC_{H,2}$ and $IC_{L,2}$ both hold while changing $\hat{w}_2$ because it was present at both sides of these expressions.

Once the seller fixes $w_2(l)$, she continues increasing the payment for message h by $\Delta w'$ up to

$$x_2(h)\theta_H - (\hat{w}_2 + \Delta w) + \delta U_{H,1}(p_2(h)) - \Delta w' = x_2(l)\theta_L - (\hat{w}_2 + \Delta w) + \delta U_{L,1}(p_2(l)).$$

At this point, the seller does not increase the payment anymore. If it were the case, the high-type buyer will send a low-type message, violating the $IC_{H,2}$. As consequence, optimality also implies $IC^*_{H,2}$. Note that $IC_{L,2}$ continues holding while the seller increases the payment $\Delta w'$ for message h -an increment in $\Delta w'$ affects only the RHS of the $IC_{L,2}$.

Step 3: $IR^*_{L,2} + IC^*_{H,2} + SMC_2 \Leftrightarrow IR^*_{L,2} + IC^*_{H,2} + IC_{L,2}$

$\Leftarrow$: we have

$$IC^*_{H,2}: \quad x_2(h)\theta_H - w_2(h) + \delta U_{H,1}(p_2(h)) = x_2(l)\theta_H - w_2(l) + \delta U_{H,1}(p_2(l)),$$

$$IC_{L,2}: \quad x_2(l)\theta_L - w_2(l) + \delta U_{L,1}(p_2(l)) \geq x_2(h)\theta_L - w_2(h) + \delta U_{L,1}(p_2(h)),$$

with equality if $q_L > 0$.

$$IR^*_{L,2}: \quad x_2(l)\theta_L - w_2(l) + \delta U_{L,1}(p_2(l)) = 0,$$

from $IR^*_{L,2}$ and by backward induction we know that $U_{L,1}(p_2(l)) = U_{L,1}(p_2(h)) = 0$. Operating with $IC^*_{H,2}$ we get:

$$w_2(h) - w_2(l) = x_2(h)\theta_H - x_2(l)\theta_H + \delta\left[U_{H,1}(p_2(h)) - U_{H,1}(p_2(l))\right],$$

plugging it into $IC_{L,2}$ and operating we get the Sequential Monotonicity Constraint (SMC_2):

$$x_2(h) + \delta\frac{U_{H,1}(p_2(h))}{\Delta\theta} \geq x_2(l) + \delta\frac{U_{H,1}(p_2(l))}{\Delta\theta}, \text{ with equality if } q_L > 0.$$

$\Rightarrow$: Starting from the SMC_2, multiplying it by $\Delta\theta$ and using $IC^*_{H,2}$ and $IR^*_{L,2}$ it is possible to recover $IC_{L,2}$. ■

A.5 Lemma 4

Lemma 36 *A mechanism with* SMC non-binding with no-learning *can be offered only when* $p_{H,3} \geq \frac{\theta_L}{\theta_H^2}\Delta\theta + \frac{\theta_L}{\theta_H}$ *and a mechanism with* SMC non-binding with learning *can be offered only when* $p_{H,3} \geq \frac{\theta_L}{\theta_H}$. *Mechanisms with* SMC binding *have no restrictions on the prior.*

Proof. A mechanism with *SMC non-binding with no-learning* requires $q_L = 0$, $\rho_{H,3} \geq \frac{\theta_L}{\theta_H}$ and $p_{H,2}(l) \geq \frac{\theta_L}{\theta_H}$. Since $q_L = 0$, from $\rho_{H,3} \geq \frac{\theta_L}{\theta_H}$ it is necessary $q_H \geq \frac{\theta_L}{\theta_H p_{H,3}}$ and from $p_{H,2}(l) \geq \frac{\theta_L}{\theta_H}$, $q_H \leq 1 - \frac{(1-p_{H,3})}{p_{H,3}}\frac{\theta_L}{\Delta\theta}$. Both conditions are satisfied when $p_{H,3} \geq \frac{\theta_L}{\theta_H^2}\Delta\theta + \frac{\theta_L}{\theta_H}$.

A mechanism with *SMC non-binding with learning* requires $q_L = 0$, $\rho_{H,3} \geq \frac{\theta_L}{\theta_H}$ and $p_{H,2}(h) \geq \frac{\theta_L}{\theta_H} > p_{H,2}(l)$. Since $q_L = 0$, from $\rho_{H,3} \geq \frac{\theta_L}{\theta_H}$, it must be that $q_H \geq \frac{\theta_L}{\theta_H p_{H,3}}$ and, from $\frac{\theta_L}{\theta_H} > p_{H,2}(l)$, it must be $q_H \geq 1 - \frac{(1-p_{H,3})}{p_{H,3}}\frac{\theta_L}{\Delta\theta}$. Since, $q_H \in [0,1]$ first condition is satisfied only when $p_{H,3} \geq \frac{\theta_L}{\theta_H}$, and the second one when $p_{H,3} < 1$.

Finally, a mechanism with *SMC binding* requires $q_L \neq 0$ or $q_L = 0$ and $\rho_{H,3} < \frac{\theta_L}{\theta_H}$. In case of *learning*, it also requires $p_{H,2}(h) \geq \tau_1^*$ (when $p_{H,3} < \frac{\theta_L}{\theta_H}$) or $p_{H,2}(l) < \frac{\theta_L}{\theta_H}$ (when $p_{H,3} \geq \frac{\theta_L}{\theta_H}$). In case of *no-learning*, $p_{H,2}(h) < \tau_1^*$ when $p_{H,3} < \frac{\theta_L}{\theta_H}$ or $p_{H,2}(l) \geq \frac{\theta_L}{\theta_H}$ when $p_{H,3} \geq \frac{\theta_L}{\theta_H}$. There is no restriction on the prior for both cases of *SMC binding*. ■

A.6 Proof of Proposition 1

Proof. It remains to get payments for both messages. We proceed by calculating them for calculating them for the optimal mechanism for every prior.

In the case of *SMC binding with no-learning*, since $x_2(l) = 1$, $w_2(l) = \theta_L$ by substituting in $IR^*_{L,2}$. From $IC^*_{H,2}$ we get $w_2(h) = \theta_L$, when substituting in it $x_2(l)$, $w_2(l)$ and using $U_{H,3}\left(p_2(h)\right) = U_{H,3}\left(p_2(l)\right)$.

In the case of *SMC not binding with learning*, $w_2(l) = 0$ by $IR^*_{L,2}$ and $x_2(l) = 0$. Since $U_{H,3}\left(p_2(l)\right) - U_{H,3}\left(p_2(l)\right) = \Delta\theta$, $w_2(h) = \theta_H - \delta\Delta\theta$ from $IC^*_{H,2}$.

Finally, when *SMC not binding with no-learning*, again $w_2(l) = 0$ by $x_2(l) = 0$. Now $U_{H,3}\left(p_2(h)\right) = U_{H,3}\left(p_2(l)\right)$, giving $w_2(h) = \theta_H$. ■

A.7 Proof of Corollary 1

Proof. Consider a message set M_2 with two possible messages $\{"take-it","leave-it"\}$, a mechanism with an allocation given by

$$x_2(m_2) = \left\{ \begin{array}{lll} 1 & if & m_2 = take-it, \\ 0 & if & m_2 = leave-it, \end{array} \right. , \; m_2 \in M_2,$$

probabilities of observing each message defined by

$$\begin{aligned} \hat{q}_i(take-it) &\equiv q_i x_2(h) + (1-q_i)x_2(l), \\ \hat{q}_i(leave-it) &\equiv 1 - \hat{q}_i(take-it), \end{aligned}$$

and posteriors $\hat{p}_{i,2}(take-it)$ and $\hat{p}_{i,2}(leave-it)$ are given by Baye's rule.

When $p_{H,3} < \frac{\theta_L}{\theta_H}$ the optimal direct selling mechanism has allocations $x_2(h) = x_2(l) = 1$, then $\hat{q}_H(take-it) = 1$, $\hat{q}_L(take-it) = 1$ and $\hat{p}_{H,2}(take-it) = p_{H,3}$. It follows that continuation values with the price posting are equal than under the direct mechanisms, i.e. $U_{i,1}(\hat{p}_2(take-it)) = U_{i,1}(p_{H,2}(h))$ for both types and $V_1(\hat{p}_2(take-it)) = V_1(p_{H,2}(h))$. Using a price $\hat{w}_2(take-it) = \theta_L$, also instant payoffs under both mechanisms are equal for every player.

When $p_{H,3} \geq \frac{\theta_L}{\theta_H}$ the optimal direct selling mechanism has payments $w_2(h) = \theta_H$ and $w_2(l) = 0$, or $w_2(h) = \theta_H - \delta\Delta\theta$ and $w_2(l) = 0$, with allocations $x_2(h) = 1$ and $x_2(l) = 0$. It follows that, $\hat{q}_H(take-it) = q_H$ and $\hat{q}_L(take-it) = q_L$ and $\hat{p}_{H,2}(take-it) = p_{H,2}(h)$ and $\hat{p}_{H,2}(leave-it) = p_{H,2}(l)$. Again, continuation values are equal for both mechanisms, i.e. $U_{i,1}(\hat{p}_2(take-it)) = U_{i,1}(p_2(h))$, $U_{i,1}(\hat{p}_2(leave-it)) = U_{i,1}(p_2(l))$, $V_1(\hat{p}_2(take-it)) = V_1(p_2(h))$ and $V_1(\hat{p}_2(leave-it)) = V_1(p_2(l))$. Using $\hat{w}_2(take-it) = w_2(h)$, also instant payoffs under both mechanisms are equal for every player.

Then, for every prior, it is possible to implement an outcome $(\hat{q}_2, \hat{p}_2, \hat{\Gamma}_2)$, where $\hat{\Gamma}_2$ is a price posting mechanism, which is payoff equivalent to the incentive efficient outcome (q_2, p_2, Γ_2) that solves (1.8) ■

A.8 Example 1: Proof

Proof. To see this notice first that the optimal mechanism under case 2 implies $q_H = 1$, $q_L = 0$. Then, following definitions at the proof of Corollary 1, if this optimal mechanism can be implemented by a price posting, it should imply $\hat{q}_H(take-it) = x_2(h)$ (i.e. $\hat{q}_H(take-it) = 1$) and $\hat{q}_L(take-it) = x_2(l)$ (i.e. $\hat{q}_L(take-it) = 1-\delta = 0.75$).

Suppose the seller proposes a price lower to θ_L. The low-type buyer makes zero payoffs at last period, and today he gets positive profits sending "$take-it$" and zero with "$leave-it$". Then, $\hat{q}_L(take-it) = 1$ which is a contradiction. This price cannot be optimal.

Suppose the seller proposes a price larger than θ_L. Now, low-type buyer strictly prefers sending "$leave-it$", otherwise he makes negative profits. Then, $\hat{q}_L(take-it) = 0$ which is a contradiction. This price cannot be optimal. ■

Suppose a price equal to θ_L. If the seller does not learn, she asks for a price equal to the low-type buyer's valuation in the second period (recall $p_{H,3} < \frac{\theta_L}{\theta_H}$), making $V_2 = \theta_L + \beta\theta_L$ which is lower than $V_2 = 2.17$. If she learns, in the second period she proposes a different price for each message observed in the first period. Since $\hat{q}_H(take-it) = 1$, $\hat{p}_{H,2}(take-it) > 0$ and $\hat{p}_{H,2}(leave-it) = 0$. Then, in case of learning, in the second period she proposes a price equal to the high-type buyer's valuation if she observes "$take-it$" in the first period and a price equal to the low-type buyer's valuation if she observes "$leave-it$". Seller gets

$$V_2 = \theta_L + \beta p_{H,3}\theta_H + \beta\left[1 - p_{H,3} - (1-p_{H,3})(1-\delta)\right]\theta_L,$$

equal to $V_2 = 1,8333$ for the parameters of the example. Then, this price cannot be optimal.

Appendix B

Appendix to Chapter 2

B.1 Seller's Sequential Problem

A general model of the seller's sequential problem has the following components.

The initial probability of facing a high-type buyer is denoted by $p_{H,T+1}$, and for a low-type buyer by $p_{L,T+1} = 1 - p_{H,T+1}$.

At every point in time r, we denote as $\bar{y}_{r+1} \equiv (\Gamma_T, m_T, ..., \Gamma_{r+1}, m_{r+1})$ to the history of past actions up to r.

The probability that the seller assigns to type i given that she observes history $\bar{y}_r$ is given by $p_{i,r} : \bar{y}_r \to [0,1]$, with $p_r \equiv (p_{L,r}(\bar{y}_r), p_{H,r}(\bar{y}_r))$.

Then at every period, the seller's strategy $\ _r$ is to choose a mechanism Γ_r given the history $\bar{y}_{r+1}$, i.e. $\ _r : \bar{y}_{r+1} \to \Upsilon$, where Υ is the space of mechanisms.

Next, the buyer observes his types i, the history and the mechanism proposed by the seller. His strategy is to send a message $m_r \in M_r$ with probability $q_{i,r}(\cdot)$, where $q_{i,r} : M_r \times \Gamma_r \times \bar{y}_{r+1} \to [0,1]$, for $i \in \{L, H\}$ and that verifies $\sum_{m_r \in M_r} q_i(m_r; \Gamma_r, \bar{y}_{r+1}) = 1$. At next period,the seller updates her beliefs and propose a new mechanism and so on.

Suppose we are at period r. Then, the seller wants to maximize her expected payoff,

$$\begin{aligned} \underset{\{q_s,p_s,\Gamma_s\}_{s=r}^{1}}{Max} \quad & \sum_{i\in\Theta} p_{i,r+1} \sum_{m_r \in M_r} q_{i,r}\left(m_r, \Gamma_r, \bar{y}_{r+1}\right)\left[w_r(m_r) + \right. \\ & \delta \sum_{i\in\Theta} p_{i,r}\left(\bar{y}_r\right) \sum_{m_{r-1}\in M_{r-1}} q_{i,r-1}\left(m_{r-1}, \Gamma_{r-1}, \bar{y}_r\right)\left[w_{r-1}(m_{r-1}) + \right. \\ & \left.\left.\delta^2 \sum_{i\in\Theta} p_{i,r-1}\left(\bar{y}_{r-1}\right) \sum_{m_{r-2}\in M_{r-2}} q_{i,r-2}\left(m_{r-2}, \Gamma_{r-2}, \bar{y}_{r-1}\right)\left[w_{r-2}(m_{r-2}) + ...\right]\right]\right], \end{aligned} \tag{B.1}$$

subject to $\{q_s, p_s, \Gamma_s\}_{s=r}^{1}$ being PBE implementable.

In order to have a PBE, the buyer's and seller's strategy must be best responses in every period. Given Γ_s, the buyer chooses his reporting strategy anticipating the future seller's beliefs (he maximizes his expected payoff (IC_r)). As response to the buyers strategy, the seller specifies an optimal sequence of mechanisms for the remainder of the game (SRC_τ). Additionally, we have the buyer's participation constraint (IR_r) and beliefs have to be consistent with Baye's Rule (BR_r). Then, the seller's problem at (B.1) is constrained to the following conditions:

$$
\begin{aligned}
IC_{i,r} \quad &: \quad \{q_{i,s}(m_s,\Gamma_s,\bar{y}_{s+1})\}_{s=r}^{1} \in \\
&\underset{\{\tilde{q}_{i,s}(m_s,\Gamma_s,\bar{y}_{s+1})\}_{s=r}^{1}}{\operatorname{argmax}} \sum_{m_r\in M_r} \tilde{q}_{i,r}(m_r,\Gamma_r,\bar{y}_{r+1})\left[u_{i,r}(m_r)+\right. \\
&\qquad \left.\delta \sum_{m_{r-1}\in M_{r-1}} \tilde{q}_{i,r-1}(m_{r-1},\Gamma_{r-1},\bar{y}_r)\left[u_{i,r-1}(m_{r-1})+...\right]\right], \quad \forall i\in\Theta, \\
IR_{i,r} \quad &: \quad \sum_{m_r\in M_r} q_{i,r}(m_r,\Gamma_r,\bar{y}_{r+1})\left[u_{i,r}(m_r)+\right. \\
&\qquad \left.\delta \sum_{m_{r-1}\in M_{r-1}} q_{i,r-1}(m_{r-1},\Gamma_{r-1},\bar{y}_r)\left[u_{i,r-1}(m_{r-1})+...\right]\right]\geq 0, \quad \forall i\in\Theta \text{ with } p_{i,r+1}>0, \\
BR_r \quad &: \quad p_{i,s}(m_s,\Gamma_s,\bar{y}_{s-1})\sum_{j\in\Theta} p_{j,s-1}q_{j,s}(m_s,\Gamma_s,\bar{y}_{s-1}) = p_{i,s-1}q_{i,s}(m_s,\Gamma_s,\bar{y}_{s-1}) \ , s=\{r,r-1,...,1\}, \\
SRC_\tau \quad &: \quad for\ all\ \tau, \tau=r-1,...,1: \\
&\qquad \underset{\{q_{\hat{s}},p_{\hat{s}},\Gamma_{\hat{s}}\}_{\hat{s}=\tau}^{1}}{Max} \sum_{i\in\Theta} p_{i,\hat{s}-1} \sum_{m_{\hat{s}}\in M_{\hat{s}}} q_{i,\hat{s}}(m_{\hat{s}},\Gamma_{\hat{s}},\bar{y}_{\hat{s}-1})\left[w_{\hat{s}}(m_{\hat{s}})+\delta\sum_{i\in\Theta} p_{i,\hat{s}}(\bar{y}_{\hat{s}})...\right], \\
&\qquad s.t.: IC_{i,\hat{s}}\ ,\ IR_{i,\hat{s}}\ ,\ BR_{\hat{s}}.
\end{aligned}
$$

where $u_{i,r}(m_r) = x_r(m_r)\theta_i - w_r(m_r)$, and assuming that the reservation utility for every type is equal zero.

Then, the seller chooses the best $\{q_s, p_s, \Gamma_s\}_{s=r}^{1}$ between all of them that are PBE implementable.

In our particular specification, at any period r the prior has all the information that the seller needs to take a decision. Then, we can write the previous problem as a recursive one where p_{r+1} is the state variable at the beginning of each period r.

Suppose we are in the last period $r=1$. Then, the seller solves,

$$
V_1(p_2) = \underset{\{q_1,\Gamma_1\}}{Max} \sum_{i\in\Theta} p_{i,2} \sum_{m_1\in M_1} q_{i,1}(m_1)\, w_1(m_1),
$$

subject to

$$
\begin{aligned}
IC_{i,1} \quad &: \quad q_{i,1}(m_1) \in \underset{\{\tilde{q}_{i,1}(m_1)\}}{\operatorname{argmax}} \sum_{m_1\in M_1} \tilde{q}_{i,1}(m_1)\, u_{i,1}(m_1) \quad \forall i\in\Theta, \\
IR_{i,1} \quad &: \quad \sum_{m_1\in M_1} q_{i,1}(m_1)\, u_{i,1}(m_1) \geq 0 \quad \forall i\in\Theta \text{ with } p_{i,2}>0.
\end{aligned}
$$

In $r=2$,

$$
\begin{aligned}
V_2(p_3) = \underset{\{q_t,p_t,\Gamma_t\}_{t=2}^{1}}{Max} \quad & \sum_{i\in\Theta} p_{i,3} \sum_{m_2\in M_2} q_{i,2}(m_2)\left[w_2(m_2)+\right. \\
& \left.\delta\sum_{i\in\Theta} p_{i,2}(m_2) \sum_{m_1\in M_1} q_{i,1}(m_1)\, w_1(m_1)\right],
\end{aligned}
$$

subject to $IC_{i,2}$, $IR_{i,2}$, BR_2 and SRC_1. Following the Principle of Optimality, previous problem can be written as

$$\begin{aligned}
V_2(p_3) &= \underset{\{q_2,p_2,\Gamma_2\}}{Max} \sum_{i\in\Theta} p_{i,3} \sum_{m_2\in M_2} q_{i,2}(m_2)\left[w_2(m_2)+\delta V_1(p_2)\right], \\
s.t. &: \\
IC_{i,2} &: \quad q_{i,2}(m_2) \in \arg\max_{\{\tilde{q}_{i,2}(m_2)\}} \sum_{m_2\in M_2} \tilde{q}_{i,2}(m_2)\left[u_{i,2}(m_2)+\delta U_{i,1}(m_2)\right] \quad \forall i\in\Theta, \\
IR_{i,2} &: \quad \sum_{m_2\in M_2} q_{i,2}(m_2)\left[u_{i,2}(m_2)+\delta U_{i,1}(m_2)\right] \geq 0 \quad \forall i\in\Theta \text{ for } i \text{ such that } p_{i,2}>0,
\end{aligned}$$

and BR_2, where $U_{i,1}(m_2) = \sum_{m_1\in M_2} q_{i,1}(m_1)\, u_{i,1}(m_1)$, i.e. the buyer's payoffs given by $IC_{i,1}$.

Recursively, let's just call $V_r(p_{r+1})$ to the problem in (B.1) at r. Then, given her beliefs, the seller solves

$$V_r(p_{r+1}) = \underset{\{q_r,p_r,\Gamma_r\}}{Max} \left[\sum_{i\in\Theta} p_{i,r+1} \sum_{m_r\in M_r} q_{i,r}(m_r)\left(w_r(m_r)+\delta V_{r-1}(p_r)\right)\right],$$

subject to $IC_{i,r}$, $IR_{i,r}$ and BR_r where

$$V_{r-1}(p_r) = \underset{\{q_{r-1},p_{r-1},\Gamma_{r-1}\}}{Max} \left[\sum_{i\in\Theta} p_{i,r} \sum_{m_{r-1}\in M_{r-1}} q_{i,r-1}(m_{r-1})\left(w_{r-1}(m_{r-1})+\delta V_{r-2}(p_{r-1})\right)\right]$$

is the $SRC(r-1)$ which is subject to $IC_{i,r-1}$, $IR_{i,r-1}$ and BR_{r-1} with $V_{r-2}(p_{r-1})$ as the $SRC(r-2)$ and so on.

B.2 Proof of Lemma 2

Proof. Suppose an incentive feasible outcome (q_r, p_r, Γ_r), where Γ_r is a direct mechanism, and with $q_L > q_H$. By the revelation principle, $q_H > 0$ and $q_L < 1$. Then all IC constraints hold with equality, i.e.

$$\begin{aligned}
IC_{H,r}: &\quad u_{H,r}(h)+\delta U_{H,r-1}(p_r(h)) = u_{H,r}(l)+\delta U_{H,r-1}(p_r(l)), \\
IC_{L,r}: &\quad u_{L,r}(l)+\delta U_{L,r-1}(p_r(l)) = u_{L,r}(h)+\delta U_{L,r-1}(p_r(h)).
\end{aligned}$$

To be incentive feasible also IR and BR have to be satisfied,

$$\begin{aligned}
IR_{H,r}: &\quad u_{H,r}(h)+\delta U_{H,r-1}(p_r(h)) \geq 0, \\
IR_{L,r}: &\quad u_{L,r}(l)+\delta U_{L,r-1}(p_r(l)) \geq 0, \\
BR_r: &\quad P_{H,r}(m_r)\sum_{j\in\Theta} p_{j,r+1}q_j(m_r) = p_{H,r+1}q_H(m_r) \quad \text{with } m_r = l,h.
\end{aligned}$$

The new outcome $(\hat{q}_r, \hat{p}_r, \Gamma_r)$ is created by renaming types. This is, $\hat{q}_H = q_L$, $\hat{q}_L = q_H$. Then $\hat{q}_H > \hat{q}_L$ as is required. The new constraints are all satisfied, with $IC_{H,r} = I\hat{C}_{L,r}$, $IC_{L,r} = I\hat{C}_{H,r}$, $IR_{H,r} = I\hat{R}_{L,r}$, $IR_{L,r} = I\hat{R}_{H,r}$ and $P_{H,r}(m_r) = \hat{P}_{L,r}(m_r)$.

Also, the seller remains indifferent, i.e.

$$\sum_{i\in\Theta}\sum_{m_r\in\{l,h\}} p_{i,r+1}q_i\left(m_r\right)\left[v_r(m_r)+\delta V_{r-1}(p_r)\right] = \sum_{i\in\Theta}\sum_{m_r\in\{l,h\}} \hat{p}_{i,r+1}\hat{q}_i\left(m_r\right)\left[v_r(m_r)+\delta V_{r-1}(\hat{p}_r)\right].$$

■

B.3 Proof of Lemma 3

Proof. In order to prove this simplification for any r we follow a similar procedure than in the static case. So, this prove has four steps:

Step 1: $IC_{H,r} + IR_{L,r} \Longrightarrow IR_{H,r}$,

First, notice that $u_{H,2}(m_2) + \delta U_{H,1}(p_2(m_2)) \geq u_{L,2}(m_2) + \delta U_{L,1}(p_2(m_2))$ $\forall m_2$ by $x_2(m_2)\theta_H - w_2(m_2) \geq x_2(m_2)\theta_L - w_2(m_2)$ and by $U_{L,1}\left(p_2\right) = 0$ $\forall p_2$ and $U_{H,1}\left(p_2\right) \geq 0$ $\forall p_2$, from solution at period $r = 1$.

From previous result, $u_{H,2}(l) + \delta U_{H,1}(p_2(l)) \geq u_{L,2}(l) + \delta U_{L,1}(p_2(l))$. By $IR_{L,2}$, $u_{L,2}(l) + \delta U_{L,1}(p_2(l)) \geq 0$. It follows that $u_{H,2}(h) + \delta U_{H,1}(p_2(h)) \geq 0$ by $IC_{H,2}$, i.e. $IR_{H,2}$ holds.

Now, suppose we are at an arbitrary period r. So, let's assume that $U_{H,r-1}(p_r(m_r)) \geq U_{L,r-1}(p_r(m_r))$ $\forall m_r$. Since $x_r(m_r)\theta_H - w_r(m_r) \geq x_r(m_r)\theta_L - w_r(m_r)$ then

$$u_{H,r}(m_r) + \delta U_{H,r-1}(p_r(m_r)) \geq u_{L,r}(m_r) + \delta U_{L,r-1}(p_r(m_r))\ \forall m_r.$$

By $IR_{L,r}$, $u_{H,r}(l)+\delta U_{H,r-1}(p_r(l)) \geq 0$. Finally, by $IC_{H,r}$, it follows that $u_{H,r}(h)+\delta U_{H,r-1}(p_r(h)) \geq 0$ and $IR_{H,r}$ holds.

Step 2: Optimality $\Longrightarrow IR^*_{L,r} + IC^*_{H,r}$,

By optimality we mean that the seller proposes an outcome (q_r, p_r, Γ_r) that maximize her profits.

From step 1, $u_{H,r}(h) + \delta U_{H,r-1}(p_r(h)) \geq u_{L,r}(l) + \delta U_{L,r-1}(p_r(l))$.

We now assume that both types start with the same payment $\hat{w}_r$, then

$$x_r(h)\theta_H - \hat{w}_r + \delta U_{H,r-1}(p_r(h)) \geq x_r(l)\theta_L - \hat{w}_r + \delta U_{L,r-1}(p_r(l)) > 0.$$

In order to improve her payoffs, the seller can increase the payment $\hat{w}_r$ asked to both types by some amount. She continues doing that up to Δw that makes $\hat{x}_r\theta_L - \hat{w}_r + \delta U_{L,r-1}(p_r(l)) - \Delta w = 0$. Fixing $w_r(l) = \hat{w}_r + \Delta w$ we get $IR_{L,r}$ binding. Note that $IC_{H,r}$ and $IC_{L,r}$ both hold while changing $\hat{w}_r$.

Once the seller fixes $w_r(l)$, she continues increasing the payment for message h by $\Delta w'$ up to

$$x_r(h)\theta_H - (\hat{w}_r + \Delta w) + \delta U_{H,r-1}(p_r(h)) - \Delta w' = x_r(l)\theta_L - (\hat{w}_r + \Delta w) + \delta U_{L,r-1}(p_r(l)).$$

At this point, the seller does not increase the payment anymore. If it were the case, the high-type buyer will send a low-type message, violating the $IC_{H,r}$. As consequence, optimality also implies $IC^*_{H,r}$. Note that $IC_{L,r}$ continues holding while the seller increases the payment $\Delta w'$ for message h -an increment in $\Delta w'$ affects only the RHS of the $IC_{L,r}$.

Step 3: $IR^*_{L,r} + IC^*_{H,r} + SMC_r \Leftrightarrow IR^*_{L,r} + IC^*_{H,r} + IC_{L,r}$
$\Leftarrow$: we have

$$IC^*_{H,r}: \quad x_r(h)\theta_H - w_r(h) + \delta U_{H,r-1}\left(p_r(h)\right) = x_r(l)\theta_H - w_r(l) + \delta U_{H,r-1}\left(p_r(l)\right),$$

$$IC_{L,r}: \quad x_r(l)\theta_L - w_r(l) + \delta U_{L,r-1}\left(p_r(l)\right) \geq x_r(h)\theta_L - w_r(h) + \delta U_{L,r-1}\left(p_r(h)\right),$$

with equality if $q_L > 0$.

$$IR^*_{L,r}: \quad x_r(l)\theta_L - w_r(l) + \delta U_{L,r-1}\left(p_r(l)\right) = 0,$$

from $IR^*_{L,r}$ and by backward induction we know that $U_{L,r-1}\left(p_r(l)\right) = U_{L,r-1}\left(p_r(h)\right) = 0$. Operating with $IC^*_{H,r}$ we get:

$$w_r(h) - w_r(l) = x_r(h)\theta_H - x_r(l)\theta_H + \delta\left[U_{H,r-1}\left(p_r(h)\right) - U_{H,r-1}\left(p_r(l)\right)\right],$$

plugging it into $IC_{L,r}$ and operating we get the Sequential Monotonicity Constraint (SMC_r):

$$x_r(h) + \delta\frac{U_{H,r-1}\left(p_r(h)\right)}{\Delta\theta} \geq x_r(l) + \delta\frac{U_{H,r-1}\left(p_r(l)\right)}{\Delta\theta}, \text{ with equality if } q_L > 0.$$

$\Rightarrow$: Starting from the SMC_r, multiplying it by $\Delta\theta$ and using $IC^*_{H,r}$ and $IR^*_{L,r}$ it is possible to recover $IC_{L,r}$. ■

B.4 Proof of Lemma 4

Proof. We proceed by induction.
From initial conditions, $\tau^*_2 = \frac{\theta_L}{\theta_H q^*_2(\tau^*_2)}$.
For $r > 2$, assume that $\tau^*_{r-1} = \frac{\theta_L}{\theta_H q^*_{r-1}(\tau^*_{r-1})}$.
From definition of τ^*_r

$$q^*_r(\tau^*_r)\tau^*_r\left(\theta_H + \delta\tilde{V}_{r-1}(1) - \delta^{r-1}\Delta\theta\right) + \left(1 - q^*_r(\tau^*_r)\tau^*_r\right)\delta\tilde{V}_{r-1}\left(\tau^*_{r-1}\right) = \theta_L + \delta\tilde{V}_{r-1}\left(\tau^*_r\right). \tag{B.2}$$

Now, let's define $\Psi_{r-1}(p)$ as

$$\Psi_{r-1}(p) \equiv q^*_{r-1}(p)p\left(\theta_H + \delta\tilde{V}_{r-1}(1)\right) + \left(1 - q^*_{r-1}(p)p\right)\delta\tilde{V}_{r-1}\left(\tau^*_{r-1}\right) - pq^*_{r-1}(p)\delta^{r-2}\Delta\theta, \quad \forall p.$$

Since $q^*_{r-1}(p) = \frac{(p-\tau^*_{r-2})}{p(1-\tau^*_{r-2})} \Rightarrow q^*_{r-1}(1) = 1$, and $\Psi_{r-1}(1) = \left(\theta_H + \delta\tilde{V}_{r-2}(1)\right) - \delta^{r-2}\Delta\theta$. By definition of $\tilde{V}_{r-1}(p)$ follows that $\tilde{V}_{r-1}(1) = \theta_H + \delta^{r-1}\tilde{V}_{r-2}(1)$. Then, we can write $\tilde{V}_{r-1}(1)$ as equal to $\Psi_{r-1}(1) + \delta^{r-2}\Delta\theta$ and (B.2) as

$$q^*_r(\tau^*_r)\tau^*_r\left(\theta_H + \delta\Psi_{r-1}(1)\right) + \left(1 - q^*_r(\tau^*_r)\tau^*_r\right)\delta\tilde{V}_{r-1}\left(\tau^*_{r-1}\right) = \theta_L + \delta\tilde{V}_{r-1}\left(\tau^*_r\right), \tag{B.3}$$

where $\tilde{V}_{r-1}\left(\tau^*_{r-1}\right)$ and $\tilde{V}_{r-1}\left(\tau^*_r\right)$ are, by definition,

$$q^*_{r-1}(\tau^*_{r-1})\tau^*_{r-1}\left(\Psi_{r-1}(1)\right) + \left(1 - q^*_{r-1}(\tau^*_{r-1})\tau^*_{r-1}\right)\delta\tilde{V}_{r-2}\left(\tau^*_{r-2}\right),$$

and

$$q^*_{r-1}(\tau^*_r)\tau^*_r\left(\Psi_{r-1}(1)\right) + \left(1 - q^*_{r-1}(\tau^*_r)\tau^*_r\right)\delta\tilde{V}_{r-2}\left(\tau^*_{r-2}\right),$$

respectively.

Since $q^*_r(\tau^*_r)\tau^*_r + \left(1 - q^*_r(\tau^*_r)\tau^*_r\right)q^*_{r-1}(\tau^*_{r-1})\tau^*_{r-1}$ is equal to $q^*_{r-1}(\tau^*_r)\tau^*_r$ and $\left(1 - q^*_{r-1}(\tau^*_{r-1})\tau^*_{r-1}\right)\left(1 - q^*_r(\tau^*_r)\tau^*_r\right)$ is equal to $\left(1 - q^*_{r-1}(\tau^*_r)\tau^*_r\right)$, the LHS of (B.3) reduces to $q^*_r(\tau^*_r)\tau^*_r\theta_H + \delta\tilde{V}_{r-1}\left(\tau^*_r\right)$. As consequence, $q^*_r(\tau^*_r)\tau^*_r\theta_H = \theta_L$, proving the last part.

Finally, using the definition of $q^*_r(\tau^*_r)$, then $\tau^*_r = \frac{\theta_L}{\theta_H}\left(1 - \tau^*_{r-1}\right) + \tau^*_{r-1}$. Suppose $\tau^*_{r-1} = \frac{\theta_L}{\theta_H}\sum_{i=0}^{r-3}\left(\frac{\Delta\theta}{\theta_H}\right)^i$, then $\tau^*_r = \frac{\theta_L}{\theta_H}\sum_{i=0}^{r-2}\left(\frac{\Delta\theta}{\theta_H}\right)^i$. ■

B.5 Proof of Lemma 5

Proof. From initial conditions, $\tau_2 = \frac{\theta_L[\theta_H + \delta\Delta\theta]}{\theta_H[\theta_L + \delta\Delta\theta]}$ and $\tau^*_2 = \frac{\theta_L}{\theta_H}$. From definition of τ_r,

$$\begin{aligned}\bar{q}_r(\tau_r)\tau_r\left(\theta_H + \delta\tilde{V}_{r-1}(1)\right) + \left(1 - \bar{q}_r(\tau_r)\tau_r\right)\delta\tilde{V}_{r-1}\left(\tau_{r-1}\right) = \\ q^*_r(\tau_r)\tau_r\left(\theta_H + \delta\tilde{V}_{r-1}(1) - \delta^{r-1}\Delta\theta\right) + \left(1 - q^*_r(\tau_r)\tau_r\right)\delta\tilde{V}_{r-1}\left(\tau^*_{r-1}\right).\end{aligned} \tag{B.4}$$

The limit of the LHS at (B.4) for $\tau_3 \to 1$ is $\theta_H + \delta\tilde{V}_2(1)$ and the one for the RHS is equal to $\theta_H + \delta\tilde{V}_2(1) - \delta^2\Delta\theta$, which is lower than the LHS. On the other hand, the limit for $\tau_3 \to \tau_2$ is $\delta\tilde{V}_2(\tau_2)$ for LHS and $q^*_3(\tau_2)\tau_2\left(\theta_H + \delta\tilde{V}_2(1) - \delta^2\Delta\theta\right) + \left(1 - q^*_3(\tau_2)\tau_2\right)\delta\tilde{V}_2\left(\tau^*_2\right)$ for RHS. From solutions for the two period case we know that,

$$\begin{aligned}&\tilde{V}_2(\tau_2) = \tau_2\theta_H + \delta\theta_L, \\ &\tilde{V}_2(1) = \theta_H + \delta\theta_H, \\ &\tilde{V}_2\left(\tau^*_2\right) = \theta_L + \delta\theta_L,\end{aligned}$$

so we get that the limit for RHS is larger than the one to the LHS and equal to $\tau_2\theta_H + \delta\theta_L + \delta^2\theta_L$. Since LHS and RHS are both continuous, then there exists at least one point such that they are equal. The derivatives of the LHS and RHS w.r.t. τ_r are constant then, the solution of (B.4) for τ_3 must be unique.

For $r > 3$, assume that the solution of (B.4) for τ_{r-1} exists and it is unique.

Taking the limit for the LHS at (B.4) for $\tau_r \to 1$, we find that it is equal to $\theta_H + \delta\tilde{V}_{r-1}(1)$, and the one for RHS is $\theta_H + \delta\tilde{V}_{r-1}(1) - \delta^{r-1}\Delta\theta$. Notice, that the limit for the LHS is larger than the one for RHS.

On the other hand, taking the limit of the LHS at (B.4) for $\tau_r \to \tau_{r-1}$, we get $\delta\tilde{V}_{r-1}(\tau_{r-1})$. For the RHS we get $q^*_r(\tau_{r-1})\tau_{r-1}\left(\theta_H + \delta\tilde{V}_{r-1}(1) - \delta^{r-1}\Delta\theta\right) + \left(1 - q^*_r(\tau_{r-1})\tau_{r-1}\right)\delta\tilde{V}_{r-1}\left(\tau^*_{r-1}\right)$, which follows the definition of $\tilde{V}_r(p)$ for $p \in [\tau_r^*, \tau_r)$ when $p = \tau_{r-1}$, i.e. $\tilde{V}_r(\tau_{r-1})$. As $\tilde{V}_r(p)$ is increasing in r, the limit for $\tau_r \to \tau_{r-1}$ of the LHS is now lower than the limit of the RHS.

Since LHS and RHS are both continuous, then there exists at least one point such that they are equal. The derivatives of the LHS and RHS w.r.t. τ_r are constant then, the solution of (B.4) for τ_r must be unique. ■

B.6 Proof of Lemma 6

Proof. Suppose $p \geq \tau_r$. Definitions of continuation values for this range of beliefs

$$\begin{aligned} \tilde{V}_r(p) &= \bar{q}_r(p)p\left(\theta_H + \delta\tilde{V}_{r-1}(1)\right) + (1-\bar{q}_r(p)p)\,\delta\tilde{V}_{r-1}(\tau_{r-1}), \\ \tilde{U}_r(p) &= (1-\bar{q}_r(p)p)\,\delta\tilde{U}_{r-1}(\tau_{r-1}). \end{aligned}$$

Applying the functional form to $\tilde{V}_{r-1}(1)$, $\tilde{V}_{r-1}(\tau_{r-1})$ and $\tilde{U}_{r-1}(\tau_{r-1})$,

$$\begin{aligned} &\tilde{V}_{r-1}(1) = \theta_H \sum_{i=0}^{r-2} \delta^i, \\ &\tilde{V}_{r-1}(\tau_{r-1}) = \tau_{r-1}\theta_H \sum_{i=0}^{r-3} \delta^i \bar{q}_{r-1-i}(\tau_{r-1}) + \delta^{r-2}\tau_{r-1}\theta_H, \\ &\tilde{U}_{r-1}(\tau_{r-1}) = 0. \end{aligned}$$

Plugging them into $\tilde{V}_r(p)$ and $\tilde{U}_r(p)$, and after some operations,

$$\begin{aligned} &\tilde{V}_r(p) = \bar{q}_r(p)p\theta_H + \bar{q}_r(p)p_{t-1}\theta_H \sum_{i=1}^{r-2} \delta^i + (1-\bar{q}_r(p)p)\,\tau_{r-1}\theta_H \sum_{i=1}^{r-2} \delta^i \bar{q}_{r-i}(\tau_{r-1}) + \delta^{r-1}p\theta_H. \\ &\tilde{U}_r(p) = 0. \end{aligned}$$

For $\tilde{V}_r(p)$, since $\bar{q}_r(p)p + (1-\bar{q}_r(p)p)\,\tau_{r-1}\bar{q}_{r-i}(\tau_{r-1}) = p\bar{q}_{r-i}(p)$, then we can write it as

$$\tilde{V}_r(p) = \bar{q}_r(p)p\theta_H + p\theta_H \sum_{i=1}^{r-2} \delta^i \bar{q}_{r-i}(p) + \delta^{r-1}p\theta_H.$$

Both, $\tilde{V}_r(p)$ and $\tilde{U}_r(p)$, follow the functional form for $p \geq \tau_r$, with $\Omega_r(p) = \varnothing$.

Suppose $p \in [\tau_r^*, \tau_r)$. Now, from definitions of continuation values,

$$\begin{aligned} \tilde{V}_r(p) &= q_r^*(p)p\left(\theta_H + \delta\tilde{V}_{r-1}(1)\right) + (1-q_r^*(p)p)\,\delta\tilde{V}_{r-1}\left(\tau_{r-1}^*\right) - pq_r^*(p)\delta^{r-1}\Delta\theta, \\ \tilde{U}_r(p) &= (1-q_r^*(p)p)\,\delta\tilde{U}_{r-1}\left(\tau_{r-1}^*\right) + \delta^{r-1}\Delta\theta. \end{aligned}$$

Again, applying the functional form to $\tilde{V}_{r-1}(1)$, $\tilde{V}_{r-1}\left(\tau^*_{r-1}\right)$ and and $\tilde{U}_{r-1}\left(\tau^*_{r-1}\right)$

$$\begin{aligned}
&\tilde{V}_{r-1}(1) = \theta_H \sum_{i=0}^{r-2} \delta^i, \\
&\tilde{V}_{r-1}\left(\tau^*_{r-1}\right) = \tau^*_{r-1}\theta_H \sum_{i=0}^{r-3} \delta^i q^*_{r-1-i}(\tau^*_{r-1}) + \delta^{r-2}\theta_L, \\
&\tilde{U}_{r-1}\left(\tau^*_{r-1}\right) = \delta^{r-2}\Delta\theta.
\end{aligned}$$

Plugging them into $\tilde{V}_r(p)$ and $\tilde{U}_r(p)$, and using $q^*_r(p)p + (1 - q^*_r(p)p)\,\tau^*_{r-1}q^*_{r-i}(\tau^*_{r-1}) = pq^*_{r-i}(p)$, we get,

$$\begin{aligned}
&\tilde{V}_r(p) = q^*_r(p)p\theta_H + p\theta_H \sum_{i=1}^{r-2} \delta^i q^*_{r-i}(p) + \delta^{r-1}\theta_L, \\
&\tilde{U}_r(p) = \delta^{r-1}\Delta\theta,
\end{aligned}$$

following the functional forms of continuation values for $p \in [\tau^*_r, \tau_r)$, again with with $\Omega_r(p) = \varnothing$.

Finally, suppose $p < \tau^*_r$. From definitions of continuation values,

$$\begin{aligned}
\tilde{V}_r(p) &= \theta_L + \delta\tilde{V}_{r-1}(p), \\
\tilde{U}_r(p) &= \theta_L + \delta\tilde{U}_{r-1}(p).
\end{aligned}$$

Applying the functional form to $\tilde{V}_{r-1}(p)$ and $\tilde{U}_{r-1}(p)$,

$$\begin{aligned}
&\tilde{V}_{r-1}(p) = \theta_L \sum_{i\in\Omega_{r-1}(p)} \delta^i + p\theta_H \sum_{i\in\bar{\Omega}_{r-1}(p)} \hat{q}_{r-1-i}(p)\delta^i + \delta^{r-2}\theta_L, \\
&\tilde{U}_{r-1}(p) = \Delta\theta \sum_{i\in\Omega_{r-1}(p)} \delta^i + \delta^{r-2}\Delta\theta,
\end{aligned}$$

and plugging them into $\tilde{V}_r(p)$ and $\tilde{U}_r(p)$,

$$\begin{aligned}
\tilde{V}_r(p) &= \theta_L \sum_{i\in\Omega_r(p)} \delta^i + p\theta_H \sum_{i\in\bar{\Omega}_r(p)} \hat{q}_{r-i}(p)\delta^i + \delta^{r-1}\theta_L, \\
\tilde{U}_r(p) &= \Delta\theta \sum_{i\in\Omega_r(p)} \delta^i + \delta^{r-1}\Delta\theta,
\end{aligned}$$

following the functional form of continuation values for $p < \tau^*_r$. ■

B.7 Proof of Lemma 7

Proof. We first show that $\tau^*_r = \tau_{r-1}\forall r \geq 2$, when $\delta = 1$.

We proceed by induction. The result is direct for $r = 2$ since by definition $\tau^*_2 = \frac{\theta_L}{\theta_H}$ and $\tau_1 = \frac{\theta_L}{\theta_H}$. It follows that $q^*_3(p) = \bar{q}_2(p)$ by their definition.

For $r > 2$, suppose $\tau^*_{r-i} = \tau_{r-1-i}$ $\forall i \in \{1, ..., r-2\}$, then, from their definitions it must be

$q^*_{r+1-i}(p) = \bar{q}_{r-i}(p)$. Additionally, from definitions of τ_r and $\tilde{V}_r(p)$, applying Lemma 6 and after some simplifications, we get

$$\tau_{r-1}\theta_H \sum_{i=0}^{r-3} \delta^i \left[\bar{q}_{r-1-i}(\tau_{r-1}) - q^*_{r-1-i}(\tau_{r-1})\right] = \delta^{r-2}\theta_L - \delta^{r-2}\tau_{r-1}\theta_H. \tag{B.5}$$

This expression, when $\delta = 1$, and using that $q^*_{r+1-i}(p) = \bar{q}_{r-i}(p)$ (due to $\tau^*_{r-i} = \tau_{r-1-i}$ by assumption) becomes $\tau_{r-1}\bar{q}_{r-1}(\tau_{r-1}) = \frac{\theta_L}{\theta_H}$. From Lemma 4, $\tau^*_r q^*_r(\tau^*_r) = \frac{\theta_L}{\theta_H}$, and since $q^*_r(p) = \bar{q}_{r-1}(p)$ (due to $\tau^*_{r-1} = \tau_{r-2}$ for $i = 1$ by assumption), it follows that $\tau^*_r = \tau_{r-1}$.

Now, let's consider the case $\delta \to 1$.

Again, we proceed by induction. For $r = 2$, $\tau^*_2 = \tau_1 = \frac{\theta_L}{\theta_H}$ and $\tau_0 = 0$ from initial conditions. For $r = 3$, $\tau^*_3 = \frac{\theta_L}{\theta_H}\left(1 + \frac{\Delta\theta}{\theta_H}\right)$ from Lemma 4 and $\tau_2 = \frac{\theta_L[\theta_H + \delta\Delta\theta]}{\theta_H[\theta_L + \delta\Delta\theta]}$ from initial conditions. Value of τ_2 is larger than τ^*_3 for $\delta < 1$. It follows that $q^*_4(p) > \bar{q}_3(p)$ by their definition.

For $r > 2$, we first show that $\frac{\partial\tau_{r-1}}{\partial\delta} < 0$. Suppose $\tau^*_{r-i} < \tau_{r-1-i}$ $\forall i \in \{1, ..., r-2\}$, then $q^*_{r-1-i}(p) > \bar{q}_{r-2-i}(p)$ from their definitions. Let's also assume that $\frac{\partial\tau_{r-1-i}}{\partial\delta} < 0$ $\forall i \in \{1, ..., r-2\}$.

Expression (B.5) can be written as

$$\tau_{r-1}\theta_H \left(1 - \sum_{i=0}^{r-3} \delta^{i-r+2} \left[q^*_{r-1-i}(\tau_{r-1}) - \bar{q}_{r-1-i}(\tau_{r-1})\right]\right) - \theta_L = 0.$$

LHS is a function of δ, τ_{r-1} and τ_{r-1-i}.[1] Let's call it $F(\delta, \tau_{r-1}, \tau_{r-1-i})$, and let's apply the implicit function theorem, i.e.

$$\frac{\partial\tau_{r-1}}{\partial\delta} = \frac{-\frac{\partial F}{\partial\delta} - \frac{\partial F}{\partial\tau_{r-2-i}}\frac{\partial\tau_{r-2-i}}{\partial\delta}}{\frac{\partial F}{\partial\tau_{r-1}}}.$$

As $(i - r + 2) < 0$ $\forall i \in \{0, ..., r-3\}$ then $\frac{\partial F}{\partial\delta} > 0$. Also, $\frac{\partial F}{\partial\tau_{r-2-i}} < 0$ (due to $\frac{\partial\bar{q}_{r-1-i}(\tau_{r-1})}{\partial\tau_{r-2-i}} < 0$) and, since $\frac{\partial\tau_{r-2-i}}{\partial\delta} < 0$ by assumption, then the numerator is negative. On the other hand, $\frac{\partial F}{\partial\tau_{r-1}} > 0$ (i.e. the denominator is positive) because, first

$$\left(1 - \sum_{i=0}^{r-3} \delta^{i-r+2} \left[q^*_{r-1-i}(\tau_{r-1}) - \bar{q}_{r-1-i}(\tau_{r-1})\right]\right)$$

has to be positive to have (B.5) equal to zero (τ_{r-1} for $r > 2$, θ_H, and θ_L are all positive) and, second

$$\frac{\partial q^*_{r-1-i}(\tau_{r-1})}{\partial\tau_{r-1}} - \frac{\partial\bar{q}_{r-1-i}(\tau_{r-1})}{\partial\tau_{r-1}} < 0,$$

by definitions of q^*_r and $\bar{q}_r$, and using the assumption $\tau^*_{r-1-i} < \tau_{r-2-i}$ and that $\tau^*_{r-2-i} < \tau^*_{r-1-i}$ from Lemma 4. It follows that $\frac{\partial\tau_{r-1}}{\partial\delta} < 0$.

[1] Although we do not write it explicitely, τ_{r-1} and τ_{r-1-i} depends on δ. By definition, $\bar{q}_{r-i}(\tau_{r-1})$ depends on $\tau_{r-1-i}(\delta)$. On the other hand, $q^*_{r-1-i}(\tau_{r-1})$ depends on τ^*_{r-1-i} which does not change with δ.

As $\tau_r^* = \tau_{r-1}$ when $\delta = 1$, then $\tau_r^* \in (\tau_{r-2}, \tau_{r-1})$ when $\delta \to 1$ by continuity. ■

B.8 Proof of Lemma 8

Proof. Since $p(h) \geq p \geq p(l)$, and since $\Omega_r(p)$ is increasing in p by definition, it follows that $|\Omega_{r-1}(p(h))| \leq |\Omega_{r-1}(p(l))|$.

When $q_L \neq 0$, the SMC_t is binding and as consequence $x_r(l) = 1 + \delta\frac{\tilde{U}_{r-1}(p(h))}{\Delta\theta} - \delta\frac{\tilde{U}_{r-1}(p(l))}{\Delta\theta}$.

Using the functional forms for continuation values,

$$x_t(l) = 1 - \sum_{i\in\Omega_{r-1}(p(l))\backslash\Omega_{r-1}(p(h))} \delta^i - \delta^{r-1}\mathbf{I}_{(p(l),p(h)]}(\tau_{r-1}).$$

In order to keep $x_r(l) \geq 0$, $\Omega_{r-1}(p(l))\backslash\Omega_{r-1}(p(h)) = \emptyset$ when $\mathbf{I}_{(p(l),p(h)]}(\tau_{r-1})$ is equal 1, and at most 1 when $\mathbf{I}_{(p(l),p(h)]}(\tau_{r-1})$ is equal 0. Then, $|\Omega_{r-1}(p(l))\backslash\Omega_{r-1}(p(h))| \leq 1$.

When $q_L = 0$, the allocation for low type message can also be $x_r(l) = 0$ ($\rho_H > \frac{\theta_L}{\theta_H}$) or $x_r(l) = 1$ ($\rho_H < \frac{\theta_L}{\theta_H}$). Under $x_r(l) = 0$ ($x_r(l) = 1$) it must be that $|\Omega_{r-1}(p(l))\backslash\Omega_{r-1}(p(h))| \leq 1$ ($|\Omega_{r-1}(p(h))| = |\Omega_{r-1}(p(l))|$), otherwise the difference between the continuation values for each message violates the SMC_r. To restore the SMC_r and make it binding, $x_r(l) = 1 + \delta\frac{\tilde{U}_{r-1}(p(h))}{\Delta\theta} - \delta\frac{\tilde{U}_{r-1}(p(l))}{\Delta\theta}$ which is the case explained above.

To see that $\Omega_{r-1}(p(l))\backslash\Omega_{r-1}(p(h)) = \max i \in \Omega_{r-1}(p(l))$ when $|\Omega_{r-1}(p(l))\backslash\Omega_{r-1}(p(h))| = 1$, let $\Omega_{r-1}(p(l)) = \{0, 1, ..., j\}$, $\Omega_{r-1}(p(h)) = \{0, 1, ..., k\}$ with $k \leq j$ for $j, k \in \{0, 1, ..., r-2\}$. Then, it must be that $k = j - 1$ and $\Omega_{r-1}(p(l))\backslash\Omega_{r-1}(p(h)) = j$. Otherwise $|\Omega_{r-1}(p(l))\backslash\Omega_{r-1}(p(h))| > 1$. ■

B.9 Proof of Lemma 9

Proof. The proof is by application of Lemma 6 for each case.

When $\Omega_{r-1}(p(l)) = \Omega_{r-1}(p(h))$, *learning* is possible only if $p(h) \geq \tau_{r-1}$ and $p(l) \in [\tau_{r-1}^*, \tau_{r-1})$. If $p(h) \geq \tau_{r-1}$ and $p(l) \geq \tau_{r-1}$ we are in *no-learning*. If $p(h) < \tau_{r-1}$ either $\Omega_{r-1}(p(l)) \neq \Omega_{r-1}(p(h))$ (contradiction) or $\Omega_{r-1}(p(l)) = \Omega_{r-1}(p(h))$ with $\tilde{U}_{r-1}(p(l)) = \tilde{U}_{r-1}(p(h))$ and we are in *no-learning* again.

When $\Omega_{r-1}(p(h)) = \Omega_{r-1}(p(l))\backslash \max\{i \in \Omega_{r-1}(p(l))\}$ (i.e. $|\Omega_{r-1}(p(l))| - |\Omega_{r-1}(p(h))| = 1$), we have $\tilde{U}_{r-1}(p(l)) > \tilde{U}_{r-1}(p(h))$. Since $p(h) \geq p \geq p(l)$ and $|\Omega_{r-1}(p(l))| - |\Omega_{r-1}(p(h))| = 1$, the set $\Omega_{r-1}(p)$ must equal to $\Omega_{r-1}(p(h))$ or to $\Omega_{r-1}(p(l))$. In both cases, it must be $p(h) < \tau_{r-1}$ and $p(l) < \tau_{r-1}$. Otherwise, since $\delta \in (\delta^*(T), 1)$, the SMC_r does not hold for any $x_r(l) \in [0, 1]$.

If $|\Omega_{r-1}(p(l))| - |\Omega_{r-1}(p(h))| > 0$, then $\tilde{U}_{r-1}(p(l)) - \tilde{U}_{r-1}(p(h)) \neq 0$. Hence, in order to have *no-learning*, it must be that $\Omega_{r-1}(p(l)) = \Omega_{r-1}(p(h))$. Additionally, it must be either $p(h), p(l) \in [\tau_{r-1}, 1]$, or $p(h), p(l) \in [0, \tau_{r-1})$. Otherwise, $\tilde{U}_{r-1}(p(l)) - \tilde{U}_{r-1}(p(h)) \neq 0$. ■

B.10 Proof of Corollary 1

Proof. Consider a message set M_r with two possible messages $\{"take-it", "leave-it"\}$, a mechanism with an allocation given by

$$x_r(m_r) = \begin{cases} 1 & if \quad m_r = take-it, \\ 0 & if \quad m_r = leave-it, \end{cases}, \ m_r \in M_r,$$

probabilities of observing each message defined by

$$\hat{q}_i(take-it) \equiv q_i x_r(h) + (1-q_i)x_r(l),$$
$$\hat{q}_i(leave-it) \equiv 1 - \hat{q}_i(take-it),$$

and the posteriors of facing a high-type buyer when observing "$take-it$", $\hat{p}(take-it)$, and the one when observing "$leave-it$", $\hat{p}(leave-it)$, are given by Baye's rule.

When $p < \tau_r^*$ the optimal direct selling mechanism has allocations $x_r(h) = x_r(l) = 1$, then, by definition, $\hat{q}_H(take-it) = 1$, $\hat{q}_L(take-it) = 1$ and $\hat{p}(take-it) = p$. It follows that continuation values with the price posting are equal than under the direct mechanisms, i.e. $U_{i,r-1}(\hat{p}(take-it)) = U_{i,r-1}(p(h))$ for both types and $V_{r-1}(\hat{p}(take-it)) = V_{r-1}(p(h))$. Using a price $\hat{w}_r(take-it) = \theta_L$, also instant payoffs under both mechanisms are equal for every player.

When $p \geq \tau_r^*$ the optimal direct selling mechanism has payments $w_r(h) = \theta_H$ and $w_r(l) = 0$, or $w_r(h) = \theta_H - \delta^{r-1}\Delta\theta$ and $w_r(l) = 0$, with allocations $x_r(h) = 1$ and $x_r(l) = 0$. It follows that, $\hat{q}_H(take-it) = q_H$ and $\hat{q}_L(take-it) = q_L$ and $\hat{p}(take-it) = p(h)$ and $\hat{p}(leave-it) = p(l)$. Again, continuation values are equal for both mechanisms, i.e. $U_{i,r-1}(\hat{p}(take-it)) = U_{i,r-1}(p(h))$, $U_{i,r-1}(\hat{p}(leave-it)) = U_{i,r-1}(p(l))$, $V_{r-1}(\hat{p}(take-it)) = V_{r-1}(p(h))$ and $V_{r-1}(\hat{p}(leave-it)) = V_{r-1}(p(l))$. Using $\hat{w}_r(take-it) = w_r(h)$, also instant payoffs under both mechanisms are equal for every player. ■

Then, for every prior, it is possible to implement an outcome $(\hat{q}_r, \hat{p}_r, \hat{\Gamma}_r)$, where $\hat{\Gamma}_r$ is a price posting mechanism, which is payoff equivalent to the incentive efficient outcome (q_r, p_r, Γ_r) that solves (2.6).

Appendix C

Appendix to Chapter 3

C.1 Proof of Proposition 3

Proof. We organize the proof as follows. First, at point i), we show that $V^U(\underline{k}) > V^P(\underline{k})$. Second, at ii), we show that $V^U(\bar{k}) < V^P(\bar{k})$. Since $V^P(k)$ and $V^U(k)$ are both continuous in k, points i) and ii) imply that there exist at least one point k^* such that $V^P(k^*) = V^U(k^*)$. Finally, at iii), we show that this crossing point is unique.

i) Suppose $k = \underline{k}$. In case of hiring at the unemployment pool, since she hires a young worker, the firm with the lowest capital stock makes positive expected profits, i.e. $V^U(\underline{k}) > 0$. On the other hand, $V^P(\underline{k}) = 0$ since the probability of successful poaching is zero.

ii) Equation (3.1) can be written as

$$V^U(k) = \frac{k\Psi(p(k)) + \delta k\alpha_{k,0}(1 - \Psi(p(k))) + \delta k\alpha_{k,1}\Psi(p(k))\phi(k)}{\big(1 - \delta\Psi(p(k))(1 - \phi(k))(1 - \delta) - \delta^2\big)}, \tag{C.1}$$

and equation (3.2) as

$$V^P(k) = \frac{1}{1-\delta}\int \left(k\alpha_{k^I,1} - k^I\right)^+ \left(\frac{\Psi(p(k^I))Prob(k^I \in B)(\frac{2-\phi(k^I)}{2})}{\int \Psi(p(x))Prob(x \in B)(\frac{2-\phi(x)}{2})dF(x)}\right) dF(k^I). \tag{C.2}$$

Suppose $k = \bar{k}$. Since the firm has the highest possible underlying capital stock, no rival firm can poach her worker tomorrow in case of going to the pool today, i.e $(1 - \phi(\bar{k})) = 0$. She has not incentives to distort her technology, and she must have chosen $(\bar{p}, 0)$. Hence, $\Psi_k(\bar{p}) = \bar{p}\alpha$ and, since

$$\bar{k}\alpha_{\bar{k},0}(1 - \Psi(p(\bar{k}))) + \alpha_{\bar{k},1}\Psi(p(\bar{k}))) = \bar{k}\alpha,$$

$$\begin{aligned} V^U(\bar{k}) &= \frac{\bar{k}\Psi(p(\bar{k})) + \delta\bar{k}\alpha}{1-\delta^2}, \\ &\leq \frac{\bar{k}\alpha}{1-\delta}. \end{aligned}$$

Since all firms were going to the pool and there were no poaching then $Prob(k^I \in B) = Prob(x \in B)$ and $\phi(k^I) = \phi(x) = 1$. Additionally, since every firm is choosing the same technology $(\bar{p})$, $\Psi(p(k^I)) = \int \Psi(p(x))dF(x)$ and $\alpha_{k^I,1} = 1$. Then, suppose that a firm with capital stock $\bar{k}$, when her old worker dies, deviates and decides to poach. Hence, she makes,

$$V^P(\bar{k}) = \frac{1}{1-\delta}\left(\int (\bar{k} - k^I)^+ dF(k^I)\right).$$

Solving we get,

$$V^P(\bar{k}) = \frac{1}{1-\delta}\left(\bar{k} - \frac{\bar{k} + \underline{k}}{2}\right).$$

To be $V^P(\bar{k}) > V^U(\bar{k})$,

$$\bar{k}\bar{p} - \bar{p}\frac{\bar{k} + \underline{k}}{2} > \bar{k}\alpha,$$

which is true for $\bar{k} > \frac{\underline{k}}{1-2\alpha}$.

iii) Suppose some $k \in (\underline{k}, \bar{k})$.

Taking the derivative of $V^U(k)$ w.r.t. k (by the Envelope Theorem we need to consider only the direct effects; technologies have been optimally chosen at $t = 0$.) we get

$$\begin{aligned}\frac{\partial}{\partial k}V^U(k) =& \frac{\Psi(p(k)) + \delta(1-\Psi(p(k)))\alpha_{k,0} + \delta\Psi(p(k))\alpha_{k,1}\phi(k)}{\left(1-\delta(1-\delta)\Psi(p(k))(1-\phi(k)) - \delta^2\right)} \\ &+ \frac{\delta k\Psi(p(k))\alpha_{k,1}\phi'(k)}{\left(1-\delta(1-\delta)\Psi(p(k))(1-\phi(k)) - \delta^2\right)} \\ &- \frac{\delta\phi'(k)\Psi(p(k))(1-\delta)k\left[\Psi(p(k)) + \delta(1-\Psi(p(k)))\alpha_{k,0}\right]}{\left(1-\delta(1-\delta)\Psi(p(k))(1-\phi(k)) - \delta^2\right)^2} \\ &- \frac{\delta\phi'(k)\Psi(p(k))(1-\delta)k\left[\delta\Psi(p(k))\alpha_{k,1}\phi(k)\right]}{\left(1-\delta(1-\delta)\Psi(p(k))(1-\phi(k)) - \delta^2\right)^2}.\end{aligned}$$

To know if the derivative is positive or negative we compare,

$$\begin{aligned}\Psi(p(k)) +& \delta(1-\Psi(p(k)))\alpha_{k,0} + \delta\Psi(p(k))\alpha_{k,1}\phi(k) + \delta k\Psi(p(k))\alpha_{k,1}\phi'(k) \geq \\ &\frac{\delta\phi'(k)\Psi(p(k))(1-\delta)k\left[\Psi(p(k)) + \delta(1-\Psi(p(k)))\alpha_{k,0}\right]}{\left(1-\delta(1-\delta)\Psi(p(k))(1-\phi(k)) - \delta^2\right)} \\ &+ \frac{\delta\phi'(k)\Psi(p(k))(1-\delta)k\left[\delta\Psi(p(k))\alpha_{k,1}\phi(k)\right]}{\left(1-\delta(1-\delta)\Psi(p(k))(1-\phi(k)) - \delta^2\right)}.\end{aligned}$$

When $\delta \to 1$, the RHS have an indeterminate form of $0/0$. To solve it, we apply L'Hopital getting,

$$\frac{\phi'(k)\Psi(p(k))k\left[\Psi(p(k)) + (1-\Psi(p(k)))\alpha_{k,0} + \Psi(p(k))\alpha_{k,1}\phi(k)\right]}{2 - \Psi(p(k))(1-\phi(k))}.$$

Comparing this result with $k\Psi(p(k))\alpha_{k,1}\phi'(k)$ (the last positive term at (**??**) when $\delta \to 1$), we get

$$\big(k\Psi(p(k))\alpha_{k,1}\phi'(k)\big)(2-\Psi(p(k))(1-\phi(k))) \geq \\ \phi'(k)\Psi(p(k))k\left[\Psi(p(k))+(1-\Psi(p(k)))\alpha_{k,0}+\Psi(p(k))\alpha_{k,1}\phi(k)\right]$$

because $\alpha_{k,1} \geq \alpha_{k,0}$ and $\alpha_{k,1} \geq \Psi(p(k))$, both due to $p(k) \geq q(k)$. It follows that $\frac{\partial}{\partial k}V^U(k) \geq 0$ when $\delta \to 1$.

On the other hand,

$$V^P(k) = \frac{1}{1-\delta}\int \left(k\alpha_{k^I,1}-k^I\right)^+ \left(\frac{\Psi(p(k^I))Prob(k^I \in B)(\frac{2-\phi(k^I)}{2})}{\int \Psi(p(x))Prob(x \in B)(\frac{2-\phi(x)}{2})dF(x)}\right) dF(k^I),$$

can be written as

$$V^P(k) = \frac{1}{1-\delta}\int_{\underline{k}}^{\Gamma(k)} \left(k\alpha_{k^I,1}-k^I\right) \left(\frac{\Psi(p(k^I))Prob(k^I \in B)(\frac{2-\phi(k^I)}{2})}{\int \Psi(p(x))Prob(x \in B)(\frac{2-\phi(x)}{2})dF(x)}\right) dF(k^I),$$

because $k\alpha_{k^I,1}-k^I > 0$ when $k^I < \Gamma(k)$.

Taking the derivative of this last expression of $V^P(k)$ w.r.t. k,[1]

$$\frac{\partial}{\partial k}V^P(k) = \frac{1}{1-\delta}\int_{\underline{k}}^{\Gamma(k)} \alpha_{k^I,1} \left(\frac{\Psi(p(k^I))Prob(k^I \in B)(\frac{2-\phi(k^I)}{2})}{\int \Psi(p(x))Prob(x \in B)(\frac{2-\phi(x)}{2})dF(x)}\right) dF(k^I).$$

Hence $\frac{\partial}{\partial k}V^P(k)$ is also positive.

Notice that, since $V^U(k^*) = V^P(k^*)$, using (C.1), (C.2) and the expression of $\frac{\partial}{\partial k}V^P(k)$ evaluated

[1]

$$\frac{\partial}{\partial k}V^P(k) = \left[\frac{1}{1-\delta}\int_{\underline{k}}^{\Gamma(k)} \alpha_{k^I,1} \left(\frac{\Psi(p(k^I))Prob(k^I \in B)(\frac{2-\phi(k^I)}{2})}{\int \Psi(p(x))Prob(x \in B)(\frac{2-\phi(x)}{2})dF(x)}\right) dF(k^I)\right] \\ + \left[\frac{k}{1-\delta}\frac{\partial}{\partial k}\int_{\underline{k}}^{\Gamma(k)} \alpha_{k^I,1} \left(\frac{\Psi(p(k^I))Prob(k^I \in B)(\frac{2-\phi(k^I)}{2})}{\int \Psi(p(x))Prob(x \in B)(\frac{2-\phi(x)}{2})dF(x)}\right) dF(k^I)\right] \\ - \left[\frac{1}{1-\delta}\frac{\partial}{\partial k}\int_{\underline{k}}^{\Gamma(k)} k^I \left(\frac{\Psi(p(k^I))Prob(k^I \in B)(\frac{2-\phi(k^I)}{2})}{\int \Psi(p(x))Prob(x \in B)(\frac{2-\phi(x)}{2})dF(x)}\right) dF(k^I)\right],$$

where the difference of last two terms is equal zero.

at k^*, we can write $\frac{\partial}{\partial k}V^U(k)$ evaluated at k^* as equal to

$$\frac{\partial}{\partial k}V^P(k^*) - \frac{1}{(1-\delta)k^*}\int_{\underline{k}}^{\Gamma(k^*)} k^I \left(\frac{\Psi(p(k^I))Prob(k^I \in B)(\frac{2-\phi(k^I)}{2})}{\int \Psi(p(x))Prob(x \in B)(\frac{2-\phi(x)}{2})dF(x)}\right) dF(k^I)$$
$$+ \frac{\delta k^* \Psi(p(k^*))\alpha_{k^*,1}\phi'(k^*)}{\left(1-\delta(1-\delta)\Psi(p(k^*))(1-\phi(k^*)) - \delta^2\right)}$$
$$- \frac{\delta\phi'(k^*)\Psi(p(k^*))(1-\delta)k^*\left[\Psi(p(k^*)) + \delta(1-\Psi(p(k^*)))\alpha_{k^*,0}\right]}{\left(1-\delta(1-\delta)\Psi(p(k^*))(1-\phi(k^*)) - \delta^2\right)^2}$$
$$- \frac{\delta\phi'(k^*)\Psi(p(k^*))(1-\delta)k^*\left[\delta\Psi(p(k^*))\alpha_{k^*,1}\phi(k^*)\right]}{\left(1-\delta(1-\delta)\Psi(p(k^*))(1-\phi(k^*)) - \delta^2\right)^2}.$$

This last expression is lower than $\frac{\partial}{\partial k}V^P(k^*)$ since, when $\delta \to 1$, $1/[(1-\delta)k^*] \to \infty$ and the difference among second to fourth line (after applying L'Hopital to solve indetermination of the form 0/0 and some simplifications) gives a positive but finite value.

It follows that at every crossing point k^* the slope of $V^P(k^*)$ is larger than the one of $V^U(k^*)$. Therefore, k^* must be unique. ■

C.2 Proof of Proposition 4

Proof. From Proposition 3, we know that all $k \geq k^*$ poaches from firms that went to the unemployment pool (from those firms with a capital stock $k \leq k^*$). Thus, firms with $k \geq k^*$ are not being poached by other poaching firms, i.e. $\phi(k)$=1. Since these poaching firms expect to poach high-skill workers and they do not have incentives to distort to avoid poaching, they must be choosing $\bar{p}$.

On the other hand, the firm with $\bar{k}$ can poach from every firm with at most a capital stock $\tilde{k}$ equal to $\bar{k}\alpha_{\tilde{k},1} - \epsilon$) with $\epsilon \to 0^+$. Hence, every firm with $k \geq \bar{k}\alpha_{\tilde{k},1}$) cannot be poached, i.e. $\phi(k)$=1. If this limit capital stock is larger or equal to k^*, all firms between k^* and this limit capital stock are chosen non distortion (as we argued above). Hence, let k_1 be the $min\left\{\bar{k}\alpha_{\tilde{k},1}, k^*\right\}$.

Suppose $k_1 = \bar{k}\alpha_{\tilde{k},1}$. Since $\phi(k) = 1$ for those $k^I \in [k_1, k^*)$,

$$V^U(k^I) = \frac{k^I\Psi(p(k^I)) + \delta k^I\alpha_{k^I,0}(1-\Psi(p(k^I))) + \delta k^I\alpha_{k^I,1}\Psi(p(k^I))}{(1-\delta^2)}.$$

Since,

$$\alpha_{k^I,0}(1-\Psi(p(k^I))) = (1-p(k^I))\alpha \text{ and,}$$
$$\alpha_{k^I,1}\Psi(p(k^I)) = p(k^I)\alpha,$$

then

$$V^U(k^I) = \frac{k^I\Psi(p(k^I)) + \delta k^I\alpha}{(1-\delta^2)},$$

and its derivative with respect to q, $k/(1-\delta^2)(- \ \alpha + (1-\alpha)) < 0$ because the assumption $\ > (1-\alpha)/\alpha$. Hence, all firms with capital stock $k \geq k_1$ do not distort and choose $\bar{p}$.

From Proposition 3, a firm with capital stock k^* is the firm with the minimum capital who poaches. From previous arguments, she must be choosing $\bar{p}$. Her better offer to a firm k^I is equal to $k^* \alpha_{k^I,1}$. Suppose that this firm with capital stock k^I is choosing the maximum distortion, i.e. $p(k^I) = \bar{p}/(1+ \ )$ (which means $\alpha_{k^I,1} = \alpha$. The better counter offer that this incumbent can make is equal to k^I. Then, every k^I such that $k^I \leq k^* \alpha$ cannot deter poaching even with the worst technology. Then, they have $\phi(k^I) = 0$.

Since $\phi(k^I) = 0$,

$$V^U(k^I) = \frac{k^I \Psi(p(k^I)) + \delta k^I \alpha (1 - p(k^I))}{(1 - \delta(1-\delta)\Psi_{k^I}(p(k^I) - \delta^2)},$$

Its derivative with respect to q is

$$\frac{k^I(- \ \alpha + (1-\alpha) + \delta \ \alpha)}{(1 - \delta(1-\delta)\Psi_{k^I}(p(k^I) - \delta^2)} - \frac{(\delta^2 - \delta)[k^I \Psi(p(k^I)) + \delta k^I \alpha (1 - p(k^I))]}{(1 - \delta(1-\delta)\Psi_{k^I}(p(k^I) - \delta^2)^2}.$$

The ratio,

$$\frac{k^I(- \ \alpha + (1-\alpha) + \delta \ \alpha)(1 - \delta(1-\delta)\Psi_{k^I}(p(k^I) - \delta^2)}{(\delta^2 - \delta)[k^I \Psi(p(k^I)) + \delta k^I \alpha (1 - p(k^I))]},$$

after taking the limit for $\delta \to 1$ (using L'Hopital to solve the indetermination 0/0), is equal to

$$\frac{(1-\alpha)(p(k^I)\alpha + q(k^I)(1-\alpha) - 2)}{(- \ \alpha + (1-\alpha))(q(k^I)(1-\alpha) + \alpha)}.$$

Previous ratio is lower than 1 (and as consequence, $\frac{\partial}{\partial q} V^U(k^I) < 0$) if $1 - \bar{p} < \frac{\alpha\gamma}{1-\alpha} - \frac{2}{\alpha}$.

Therefore, when $1 - \bar{p} < \frac{\alpha\gamma}{1-\alpha} - \frac{2}{\alpha}$, active firms with $k \leq \alpha k$ find optimal not to distort at all, choosing $\bar{p}$.

On the other hand, when $1 - \bar{p} \geq \frac{\alpha\gamma}{1-\alpha} - \frac{2}{\alpha}$,active firms with $k \leq \alpha k$ choose some positive distortion, i.e. $p(k) \leq \bar{p}$.

Remaining firms that go to the unemployment pool for a worker (those $k \in (\alpha k, k_1)$) are able to stop poaching by distorting their technology. So, they are choosing some $p(k) \leq \bar{p}$. ■

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